$$\sigma = \frac{P}{A}$$

$$G = \frac{E}{2(1+\nu)}$$

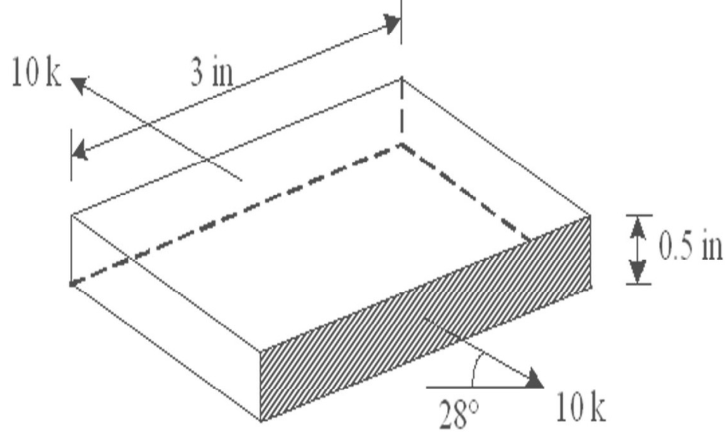
$$\varepsilon = \frac{\Delta L}{L}$$

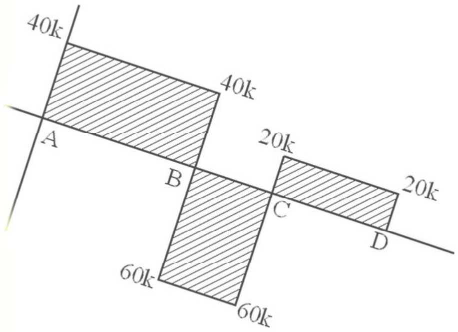
$$\sigma = \frac{P}{A}$$

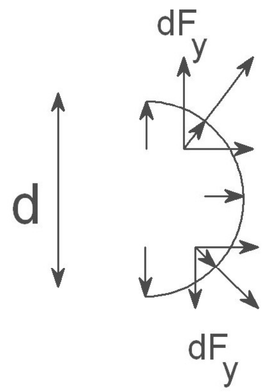
$$G = \frac{E}{2(1+\nu)}$$

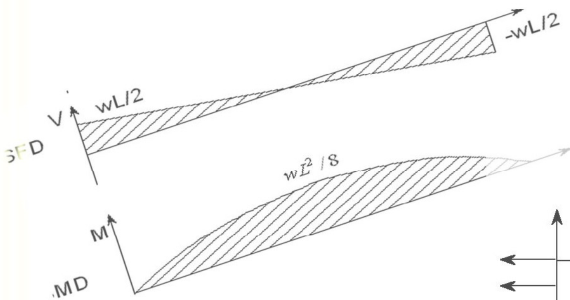
$$\varepsilon = \frac{\Delta L}{L}$$

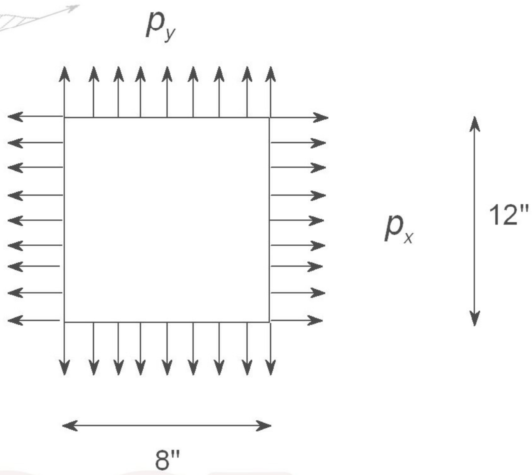
$$\sigma = \frac{P}{A}$$











MECHANICS OF MATERIALS

Textbook for a fundamental mechanics course

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Chapter 2

Stress and Strain

Goal: The learning objectives in this chapter are as follows:

1. Description of *intensities of force (stress)* and *deformation (strain)*
2. Quantification of stress and strain
3. Physical and mathematical description of the types of stresses and strains
4. Basic application of these measures to simple problems
5. Usage of these measures in design problems

2 Stress and Strain

2.1 Introduction

The previous chapter dealt with the fundamentals of Statics as used in a course on mechanics of materials. The basics of finding the forces on members of a truss or a frame, the forces at the connections etc. were described. This chapter takes the next step of describing the use of finding the forces in the structure.

The main objective of this chapter is to clearly outline the concepts of *stress* and *strain*. Stress is intensity of force and strain is intensity of displacement. In the analysis of any structure, forces and displacements are inadequate quantities in order to come up with an efficient and structurally safe design. It will be seen shortly, how important the knowledge of stress and strain is to an engineer. This chapter outlines the different types of stresses and strains, namely the *axial* and *shear* stress.

There are three different kinds of problems that a reader can expect in any engineering environment and they are addressed in the form of examples in this and other chapters.

1. The first type of problem is the straight forward one in which all the data pertaining to the geometry of the structure and the loads are completely specified. The question would involve the direct determination of the quantities such as stress and strain using appropriate principles and equations.
2. The second type of problem is of an indirect type. In such problems some of the data pertaining to either the geometry or forces is missing but stress or strain data is given. The question would be to find the missing geometry or force information. The reader should keep in mind that while these problems might appear more difficult on paper, if the principles and steps are followed as in the first type of problem the solution process becomes relatively simpler.
3. The third type of problem is design based. In such questions, the allowable values of stress and strain quantities and the forces acting on the structure are given. The unknown is the geometry of the sections such the cross section details.

2.2 Stress

Stress is such a commonly used term in one's daily life that nobody really thinks about it when using it. A good beginning to try to understand the meaning of stress is to verbalize your definition of stress and write it down in the space below.

To Do: Define stress in your own words

Definition: *Stress* can be broadly defined as the *intensity of force.* Stress is the internal distribution of forces within a body that balances and reacts to the loads applied to it. Mathematically a stress is calculated as a force divided by an area on which the force is acting.

$$\text{Stress} = \frac{\text{Force}}{\text{Area}}$$

Relating this to your own daily experience, you do not feel stressed unless you are faced with numerous deadlines in a short period. Stress in your body is something you can only experience and describe. Similarly, stress in any object is a hypothetical quantity that is internal to it. Force is real and definite while stress is something that is calculated based on many factors that will be shortly described.

Types of Stresses

There are two basic types of stresses – based on the physical action – that are used through out this subject. They are:
1. Axial or Normal Stress
2. Shear Stress

The basic steps involved in the definition and computation of any stress are
1. Isolating an object in equilibrium under a set of external forces
2. Sectioning (cutting a slice) the object into two independent parts – to expose an area on which the stress is desired
3. Using equilibrium principles to compute the internal force on the area
4. Using principles described in this chapter – calculate the stresses from the internal force and the area resisting the internal force

2.2.1 Axial or Normal Stress

Consider a circular rod subjected to two equal and opposite forces acting along the axial direction, as shown in Figure 2.1. The axial direction has a special meaning in mechanics of materials. Consider a line drawn through the centroid of areas of the cross-section at the two ends of an object. This line is called the *axial direction* as shown in Figure 2.1.

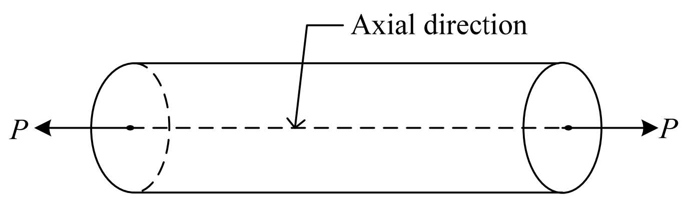


Figure 2.1: Definition of the axial direction

Now, cut the rod into two parts by passing a section that exposes an area that is *perpendicular* to the axial direction as shown. By considering the equilibrium of either section, it can be readily seen that the *internal forces* P_1 that act on either section must be equal and opposite of the external force in that particular section. This is illustrated in the drawing shown below. Thus it can be seen that the internal force P_1 in each section is equal in magnitude and is acting in the direction opposite of the external force P.

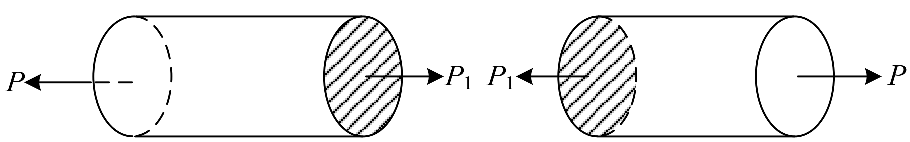
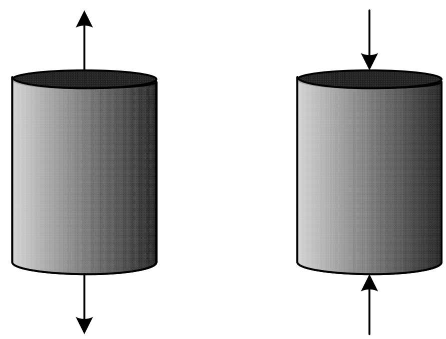


Figure 2.2: Method of sections to expose an internal area and the internal force

These internal forces cause stresses in the rod. Since the force causing this stress is acting in axial direction or in other words, the force is acting normal (synonymous to perpendicular) to the cross-section, this stress is termed as *axial* or *normal* stress. The Greek letter σ is typically used to identify the normal stress. The *axial* or *normal* stress can be expressed in equation form as follows.

$$\sigma = \frac{P_1}{A} \qquad \text{Eqn (2.1)}$$

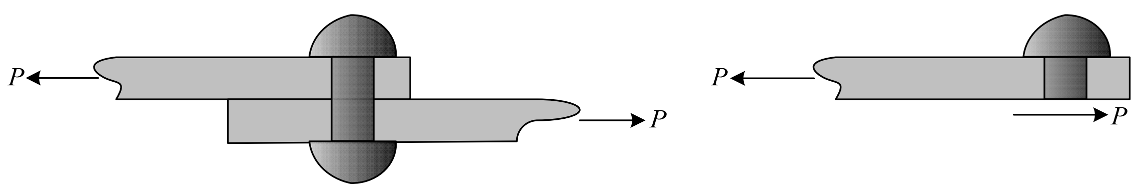
There are two types of axial or normal stress that an internal area can experience. These depend on the direction of the two equal and opposite external forces. If the two external forces act in order to pull the rod apart then the axial stress is *tensile* in nature (*rod is in tension*). On the other hand if the two external forces tend to compress the rod, the axial stress experienced by an internal area perpendicular to this force is *compressive.* The Figure shown below illustrates the difference between *tensile* and *compressive normal stresses*.



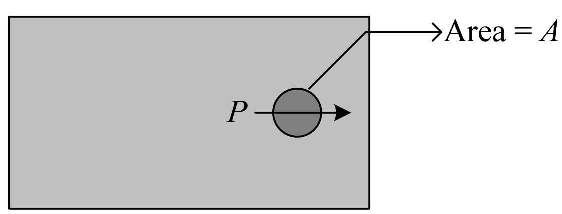
(a) Rod in tenion (b) Rod in compression
Figure 2.3: Difference between tensile and compressive stress

2.2.2 Shear Stress

The second basic type of stress is called the *shear* stress – based on the shearing action it causes to the internal area. Consider two plates connected by a single bolt and being pulled apart by two equal and opposite forces *P* as shown in the Figure 2.4. The area of the bolt in between the two plates experiences a shearing action and the stress the bolt area experience is called *shear stress.*



(a) Elevation



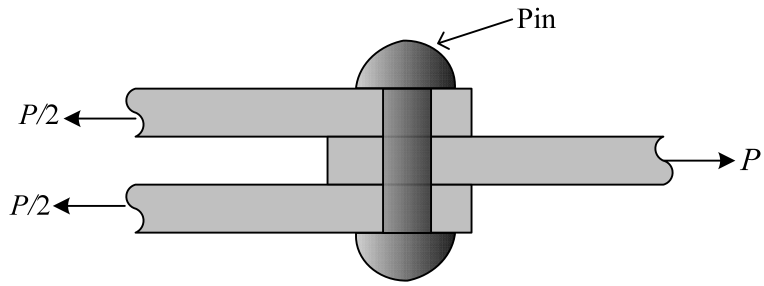
(b) Plan showing bottom side of the plate
Figure 2.4: Pin in single shear

The internal force on the bolt is computed by taking a section in between the two plates as shown in Figure 2.4. This process exposes the internal force. The internal force itself is computed by considering the force equilibrium of the section. Here the internal force in the bolt is also *P*. The shear stress on the area (represented by the Greek letter τ) is given as

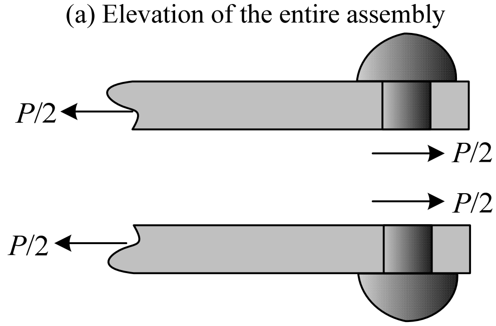
$$\tau = \frac{P}{A} \qquad \text{Eqn (2.2)}$$

Single Shear: Figure 2.4 shows the example of a pin in single shear. The action implies that the entire force is taken up by one sectional area of the bolt (the area in between the two plates). Equation 2.2 shown above is used to compute this stress.

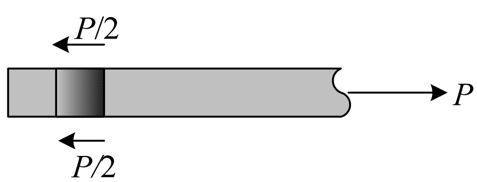
Double Shear: Figure 2.5 shows the transfer of a force from one plate to two plates, all connected by a single bolt.



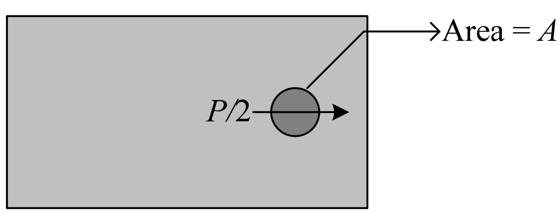
(a) Elevation of the entire assembly



(b) Section showing top and bottom plate



(c) Section of middle plate



(d) Plan showing the bottom side of the plate

Figure 2.5: Pin in Double Shear

Due to nature of the connection, there are two areas of the bolt that are effective in carrying the force *P*. Figure 2.5(b) shows a section showing the top and bottom plates (each taking up the half the force *P*). The shear stress in the bolt computed from this section is

$$\tau = \frac{\text{Force}}{\text{Area}} = \frac{P/2}{A} = \frac{P}{2A} \qquad \text{Eqn (2.3)}$$

Another way to look at it is by considering the free body diagram of the middle plate, shown in Figure 2.5(c). In contrast to the top and bottom plate here, there are two areas of the bolt that are resisting the force. The shear stress in the bolts can be computed as follows

$$\tau = \frac{\text{Force}}{\text{Area}} = \frac{P}{2A} \qquad \text{Eqn (2.4)}$$

Considering either the top or the middle plate, it is seen that the shear stress is the same. This form of action of the bolt is called *double shear*.

2.2.3 Bearing Stress

The term bearing stress is used to denote a stress that is:

- Compressive in nature
- Perpendicular to the surface
- Occurs between two surfaces

Two cases of bearing stress are presented here as an illustration. The first case represents the transfer of a compressive force between two bodies, while the second case represents the bearing stress that occurs between a bolt and a plate.

Case 1: Bearing between a column and a pedestal

Figure 2.6 shows a column transmitting a force to a pedestal. The column transmits an axial force P and has a cross section area as shown. The bearing stress on the pedestal acts uniformly over the entire bearing area which is also shown in Figure 2.6

$$\sigma = \frac{\text{Force}}{\text{Bearing Area}} = \frac{P}{b \times d} \qquad \text{Eqn (2.5)}$$

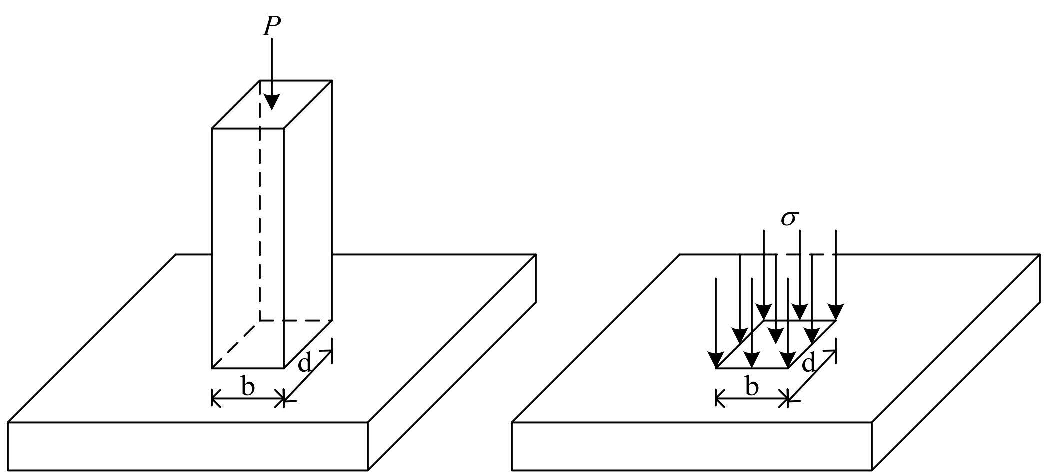


Figure 2.6: Transfer of load from column to pedestal

Case 2: Bearing between bolt and a plate

Figure 2.7: Bolt in shear and bearing

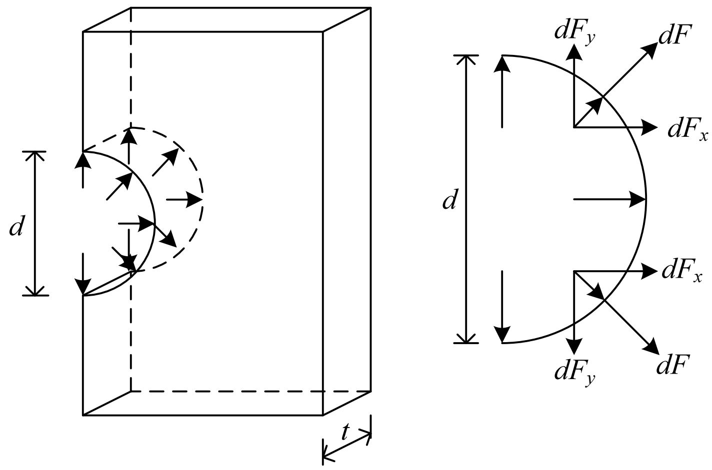


Figure 2.8: Actual bearing stress experience by the hole

When the circular bolt bears against the surface of the hole, the stress theoretically acts along the radial direction over the entire semi-circular area as seen in Figure 2.8. However, the vertical components of the force acting on each symmetric area cancel out and only the horizontal components add up to the external force *P*. In order to calculate the bearing stress one can alternatively take the projection of the cylindrical area onto a vertical surface as shown in Figure 2.9.

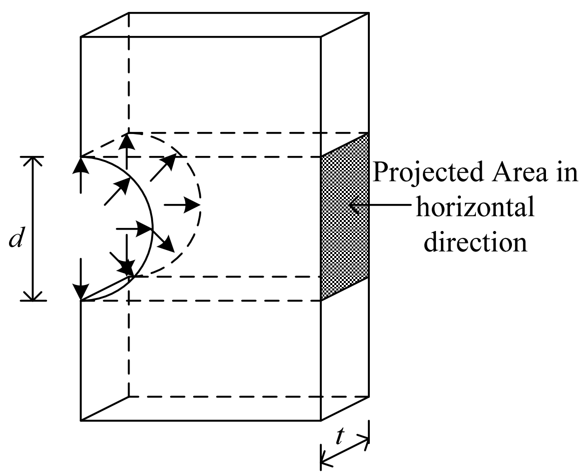


Figure 2.9: Projected area for stress computation

The bearing stress on the surface can be computed using the following expression

$$\sigma = \frac{\text{Force}}{\text{Projected Area}} = \frac{P}{d_{hole} \times t_{plate}} \qquad \text{Eqn (2.6)}$$

2.2.4 Practical Units of Stress

As a student's training to be an engineer progresses, it will be realized that units of any physical measure are what gives it a meaning. Here, since stress is a force unit divided by an area unit, the *basic unit of stress* is

- Pounds per square feet - *psf* - in U.S. Customary units
- Newton per square meter - N/m^2 - in SI Units

U.S. Customary Units:
Since *psf* is a rather small quantity (one pound acting on a square foot of area) there are other derivatives of this unit which are more frequently used. The list below gives some frequently used units for stress

- *psi* - Pound per square inch
- *ksi* - kilo Pound (1000 *lbs*.) per square inch
- *msi* - million Pounds (1,000,000 *lbs*.) per square inch

SI Units:
In SI Units, one Pascal (*Pa*) defined as a Newton force acting on one square meter of area.

To Do: The stress your weight exerts per square meter of floor area = ____ *Pa*.

From the above numerical value it can be clearly seen that a *Pa* is not a practical unit to use. A *Pa* is a very small stress value. Even normal stress values would run into seven to eight digit numbers. Some of the frequently used units and its multiples for stress in SI units are given below

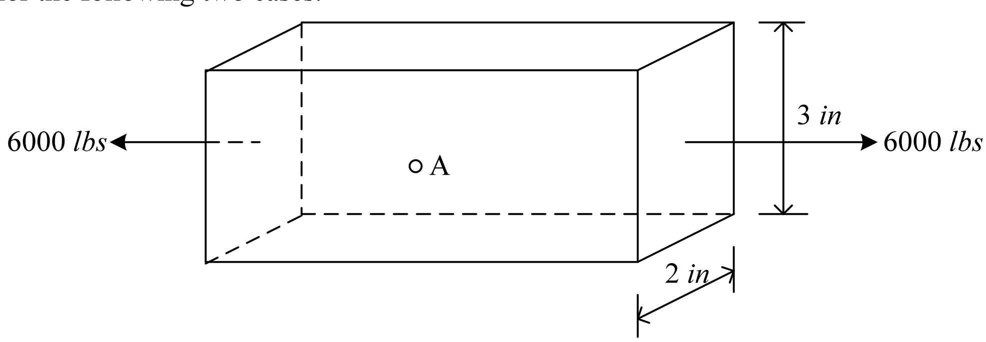
- *kPa* - kilo Pascal - 10^3 N/m^2 = 1 kN/m^2
- *MPa* - Mega Pascal - 10^6 N/m^2 = 1 N/mm^2
- *GPa* - Giga Pascal - 10^9 N/m^2 = 1000 *MPa*

2.2.5 Dependence of Stress on Force and Area

The example shown below illustrates an interesting dependence of stress on the force and an area. It is illustrated in the example that while the internal forces in the section of the rod and the physical rod itself is identical, different cross-section areas of the rod experience different kinds of stress. The student is encouraged to think and understand this example to get a better insight into the concept of stress.

Example 2-1: Illustration of the Concept of Area and Force and Stress

Problem Statement: A rectangular bar of dimensions $2\ in \times 3\ in \times 24\ in$ is subjected to an axial force of 6000 *lbs* as shown in the figure. At point 'A' on the bar as shown, determine the stresses at 'A' for the following two cases.

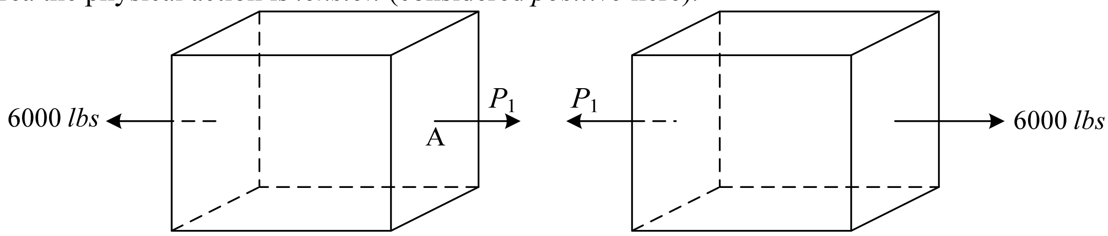


Required: 1) Find the stresses at 'A', if the cross sectional area is cut perpendicular to the axial direction.
2) Find the stresses at 'A', if the cross sectional area is cut at an angle of 30 degrees to the horizontal as shown.

Solution: Part 1

Step 1: Section the bar at point A and determine the net internal force acting on the section cut from the force equilibrium equation.

Sign Convention: If the internal forces on the sectioned area are shown acting away from the area the physical action is *tension* (considered *positive* here).



In the above figure the two sections can be defined as the left section and the right section. From the left section free body diagram the following equilibrium equation can be written

$$\sum \vec{F}_x = 0: \ \therefore -6000 + P_1 = 0$$

$P_1 = +6000\ lbs.$ (+ve hence tension)

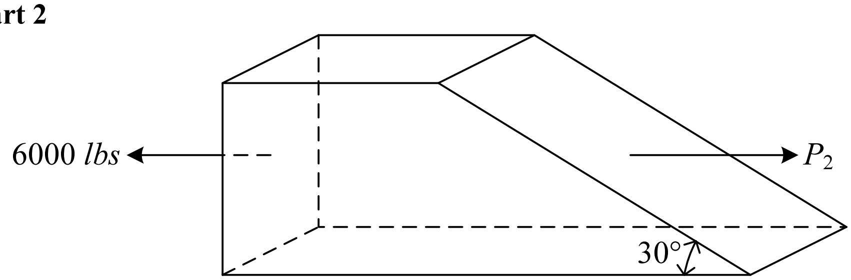
Step 2: Determine the area resisting the force

$A_1 = 3 \times 2 = 6\ in^2$

Step 3: Since the force P_1 *acts perpendicular and away* from the area, there is only one stress acting on this area – which is the axial or normal stress. Hence the stress at 'A' is given as

$$\sigma = \frac{6000\ lbs.}{6\ in^2} = 1000\ psi \text{ (tensile)}$$

Solution: Part 2



Step 1: Section the rod at point 'A' and determine the net internal force acting on the section cut from equilibrium

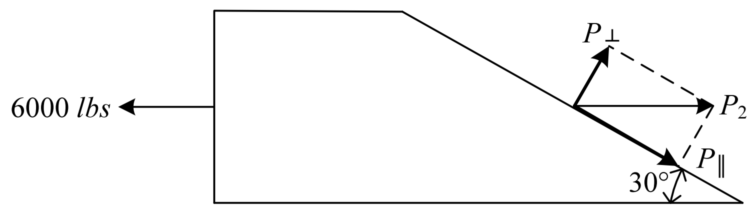
$$\sum \vec{F}_x = 0: \ \therefore -6000 + P_2 = 0$$

$P_2 = +6000\ lbs.$ (acting outwards as shown)

Step 2: Determine the area resisting the force. Here the actual area is the inclined area, which is greater than the visible cross section of 6 in^2 of the bar.

$$A_2 = \frac{3 \times 2}{\cos 60°} = 12\ in^2$$

Step 3: Since the force P_2 *acts neither perpendicular nor parallel* to the area, this force is resolved into components perpendicular $P_\perp$ (to get the normal stress) and parallel P_P (to get the shear stress).



$$P_\perp = P_2 \sin 30 = 6000 \sin 30° = 3000\ lbs$$
$$P_\parallel = P_2 \cos 30 = 6000 \cos 30° = 5196.15\ lbs$$

Step 4: Determine the stress due to the parallel (shear stress) and perpendicular (normal stress) components of the forces acting on the area

$$\sigma = \frac{P_\perp}{A_2} = \frac{3000\ lbs.}{12\ in^2} = 250\ psi\ \text{(tensile)}$$

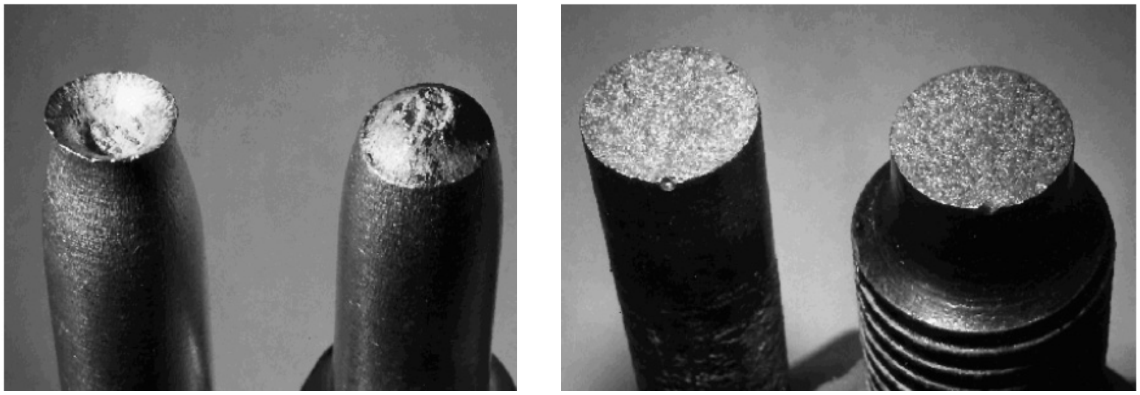
$$\tau = \frac{P_P}{A_2} = \frac{5196.15\ lbs.}{12\ in^2} = 433\ psi\ \text{(shear)}$$

Note: The point A is the same, but the stresses acting at the point depend on how the area is sectioned. It could be either normal or a combination of normal and shear. Hence, we never refer to stress at a point, rather a point on a given area. The significance of this concept will be illustrated in a later chapter of stress transformations.

Q: From Example 2-1 an important question may arise in your mind. Why should we be concerned with the determination of stress values on two different planes? A detailed answer to this will be evident as you go through related topics that appear later in the course. However, a brief explanation is given at this point.

Any material typically has different *failure stress* values in tension, compression and shear. Simplistically, this can be thought of as an experimentally determined limiting stress value, which when exceeded causes failure or fracture in a material. Ductile materials such as aluminum and steel tend to have a lower failure stress value in shear as compared to tension. On the other hand, brittle materials such as cast iron, chalk, concrete etc. tend to have a lower tensile failure stress value compared to shear.

Figure 2.10 shows the failure/fracture of two geometrically identical specimen; one of which is made of aluminum which is a ductile material (tends to fail in shear first as it has a lower shear failure stress value) while the second specimen is made of cast iron (tends to fail in tension first).



(a) Cup and cone failure of Aluminum (b) Brittle failure in cast iron

Figure 2.10: Failure modes in a ductile (Aluminum) and brittle (Cast Iron) specimen in tension

When a pure axial force is applied to the rod, note that the failure mode for the aluminum specimen has a diagonal shape, while the cast iron specimen fractures along a plane perpendicular to the axial force. This clearly indicates that the shear stress on an inclined plane of the aluminum failed before a normal stress on a plane perpendicular to the normal force reached its failure value. The case is reversed for the brittle specimen, where the normal stress on a plane perpendicular to the axial force reached its limiting value before the shear stress on an inclined plane could reach its limiting value. This concept along with more details associated with this topic will be addressed in a subsequent chapter on stress transformations.

2.2.6 Force, Traction and Stress

So far we have seen the relationship between a force and two kinds of stresses, namely the normal and shear stress. It has been shown that a stress is either parallel or perpendicular to an area. There are three terms that are frequently used in this subject. They are *force, traction vector, stress tensor.* The distinction between these terms is described in Figure 2.11.

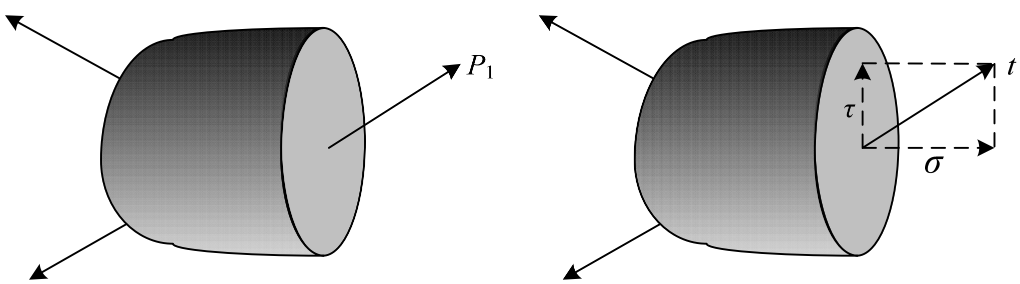


Figure 2.11: Internal force, Traction vector and Stress components

Force: The net internal force on the area sectioned in the body, is calculated from equilibrium consideration of each section. This resultant force is a vector quantity. In Figure 2.11 this is depicted by the vector P_1.

Traction (Stress) Vector: This quantity has the units of stress but has the characteristics of a vector. It is defined mathematically as

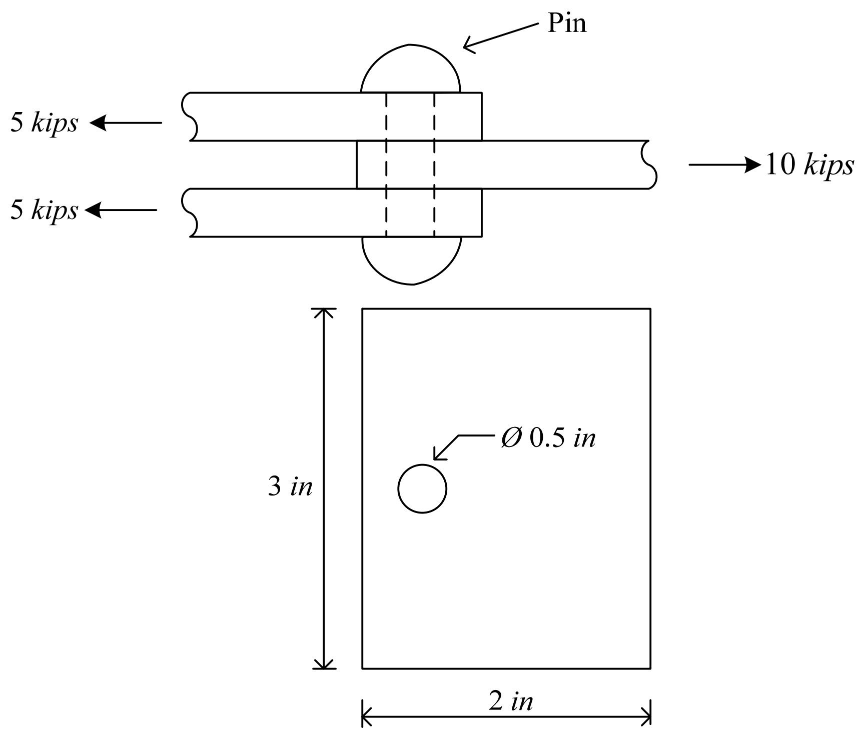
$$\vec{t} = \frac{\vec{P}}{\text{Area}}$$

This quantity is in the direction of the resultant force and has the units of stress, hence it is also known as the stress vector. In this subject this term is not used frequently.

Stress: Once the internal resultant force P_1 is resolved into parallel and perpendicular components and these components are divided by the cross-section area in order to get the two stress components that have been discussed so far. Figure 2.11 illustrates the stress terms.

Example 2-2: Tensile stress, shear stress and bearing stress

Problem Statement: A pin is used to transmit the force from the middle plate to the two plates at the top and bottom as shown. Determine the different kinds of stresses acting on the plate and the bolt in the process of transmitting this force. The thickness of each of the plates is 0.5 *inches*.



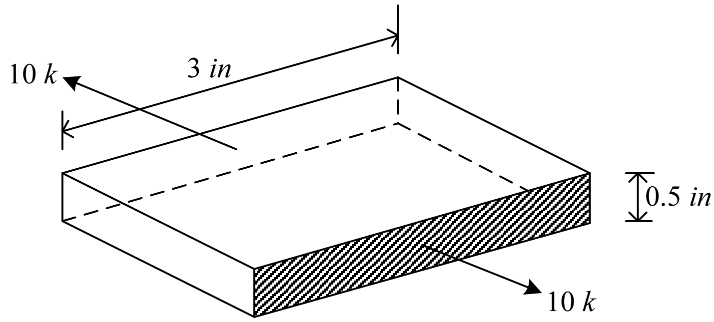
Required: Find the following stresses

a) Stress in the plate at a section away from the home
b) Stress in the plate at a section taken at the center of hole
c) Shear Stress in the bolt
d) Bearing stress on the top plate
e) Bearing stress on the middle plate

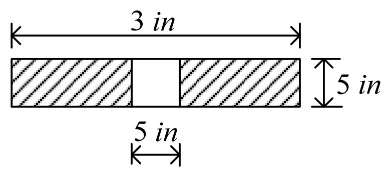
Solution:

Step 1: Determine the nature and magnitude of stress in the plate at a section away from the hole. This stress is tensile in nature. In order to determine the stress the following section FBD is used, the internal force determined and the stress computed by dividing the force by the area resisting the force.

$$\sigma = \frac{10\ k}{3 \times 0.5\ in^2} = 6.67\ ksi$$



Step 2: Stress at a section of the plate through the center of the hole: This concept is important as the area at this section is the least. Consequently, the stress at this section is the maximum stress.



$$\sigma = \frac{\text{Force}}{A_{net}}$$

$$A_{net} = A_{gross} - d_{hole} \times t_p$$

$$= 3 \times 0.5 - 0.5 \times 0.5 = 1.25\ in^2$$

$$\sigma = \frac{10\ k}{1.25\ in^2} = 8\ ksi \text{ (Tensile)}$$

Step 3: Shear Stress on the bolt. Since two areas of the bolt resist the force this bolt is in double shear

$$\tau = \frac{\text{Force}}{2A} = \frac{10\ k}{2 \times \pi \times r^2}$$

$$= \frac{10\ k}{2 \times \pi \times 0.25^2} = 25.46\ ksi$$

Step 4: Bearing stress on the middle plate

$$\sigma = \frac{\text{Force}}{\text{Bearing Area}} = \frac{\text{Force}}{d_{bolt} \times t_{plate}}$$

$$= \frac{10\ k}{0.5\ in \times 0.5\ in} = 40\ ksi$$

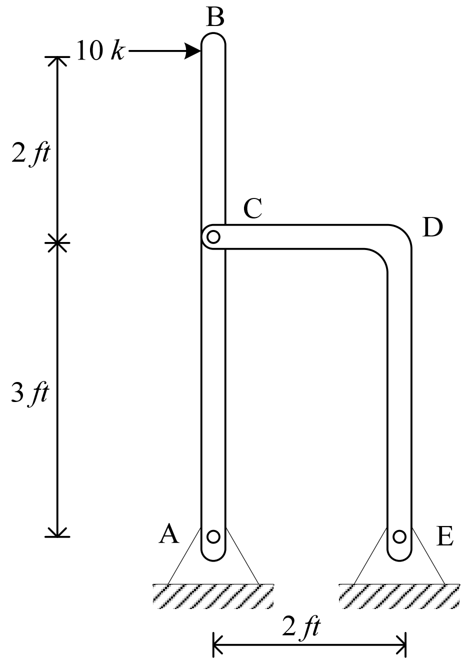
Step 5: Bearing stress on the top and bottom plate

$$\sigma = \frac{\text{Force}}{\text{Bearing Area}} = \frac{\text{Force}}{d_{bolt} \times t_{plate}}$$

$$= \frac{5\ k}{0.5\ in \times 0.5\ in} = 20\ ksi$$

Example 2.3: Design problem using a frame problem

Problem Statement: Given that the diameter of the bolt at C connecting the two members of the frame is 0.5 *in*, determine the shear stress acting in the bolt.



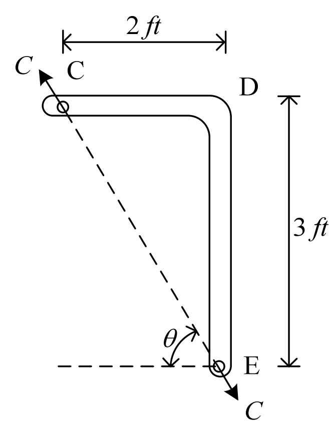
Required: Find the shear stress in the bolt.

Solution:

Step 1: Determine the force acting at C. For this a frame analysis is required. From the figure it can be seen that member ABC is a multi-force member while member CDE is a 2-force member.
Equilibrium of a 2F member: Forces are acting at C and E only. Hence, the resultant force goes along the line joining C and E. From geometry, the angle that this line makes and hence the resultant, is as shown in the figure.

$$\tan\theta = \frac{3}{2}$$

$$\theta = 56.31°$$



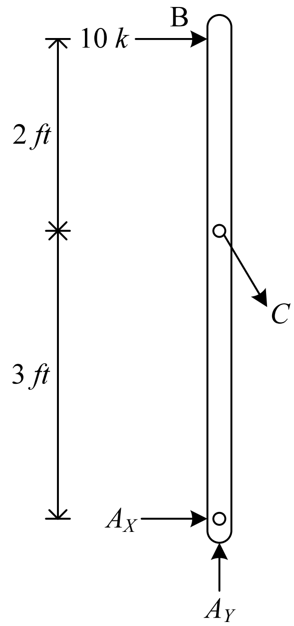
Step 2: Draw the free body diagram of the multi force member ABC, determine the force at C. The force at C is the force acting in the bolt.

$\sum M_A = 0$

$\therefore -(C\sin\theta)\times 3 - 10\times 5 = 0$

$\therefore -(C\sin 56.31°)\times 3 - 10\times 5 = 0$

$C = -20.03\ kips$



Step 3: The shear stress in the bolt is then

$$\tau = \frac{B}{A_{bolt}} = \frac{20.03\ kips}{\pi(0.5)^2/4} = 102\ ksi$$

2.3 Deformation

So far we have discussed the 'strength' aspects of mechanics of materials. As outlined in the introduction, the second fundamental characteristic of this course is the stiffness aspect. When an object is subjected to a set of external forces and moments it changes in shape. In other words, the object deforms. A measure of the stiffness of an object is the amount of deformation it experiences. Lesser the deformation, greater is the stiffness of the body.

Two basic types of deformation are considered in the following sections. They are,
1. Axial Deformation
2. Shear Deformation

These two types of deformations and the different measures that exist are described in detail below.

2.3.1 Axial Displacement, Elongation and Axial Strain

We will consider bodies subjected to an axial force in this chapter. There are three fundamentally different measures to describe the deformation of any body. The three terms can be briefly defined as follows:

1. **Displacement** → of a *point*. A specific point is said to displace when the object deforms. The displacement is measured in feet, meters, inches, mm etc. Displacement can be positive or negative. A positive displacement indicates that the point has moved in the positive axis direction. This term will be denoted with the letter *u.* For example u_A denotes the displacement of point A.
   - **Sign Convention:** + ve indicates a displacement in the + ve axis direction.

- **Units:** Length units (Example: *ft*. in US units or *mm* in SI Units).

2. **Elongation** → of a *region*. This measures the stretch of a region in the bar. It can be visualized as the change in distance between two points after a deformation has occurred. This term will be denoted by the Greek letter Δ. For example Δ_{AB} denotes the elongation of region AB.
   - **Sign Convention:** +ve indicates elongation of the bar whereas -ve indicates compression of the bar.
   - **Units:** Length units (Example: *ft*. in US units / *mm* in SI Units).
3. **Axial Strain** → in the *region*. The axial strain in a region is defined as the *intensity of elongation.* It is defined as the elongation of region divided by its original length. This term will be denoted using the Greek letter ε. For example, ε_{AB} denotes the axial strain in region AB. The axial strain is mathematically defined as follows

$$\varepsilon_{AB} = \frac{\Delta_{AB}}{L_{AB}} \qquad \text{Eqn (2.6)}$$

   - **Sign Convention:** +ve indicates elongation of the bar / -ve indicates compression of the bar.
   - **Units:** No units. However, (*in. /in.*), (*mm/mm*) etc. are frequently used.

The following example clearly illustrates the difference between the three measures.
Consider a circular rod subjected to an axial force as shown in Figure 2.12. Furthermore, consider two points on the rod A and B that are half *inch* apart. Point A is one *inch* from the origin O. After elongation the points now are located as shown.

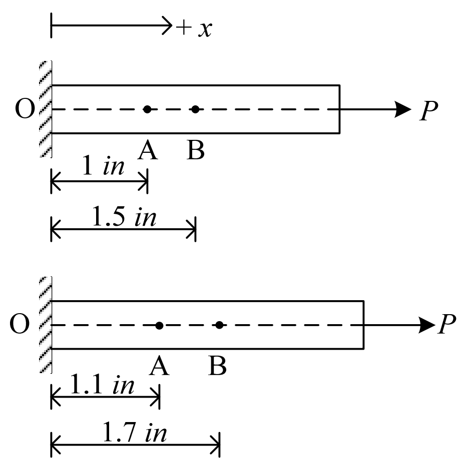


Figure 2.12: Axial displacement, elongation and strain

Original locations:
- Point A: 1 *inch* from the origin O
- Point B: 1.5 *inches* from the origin O

After deformation:
- Point A: 1.1 *inches* from the origin O
- Point B: 1.7 *inches* from the origin O

Various deformation measures are defined as follows.
Displacement of points:
- u_A = 1.1 – 1.0 = +0.1 *in*
- u_B = 1.7 – 1.5 = +0.2 *in*

Elongation of AB:

- $\Delta_{AB} = l_{A'B'} - l_{AB} = 0.6 - 0.5 = +0.1\ in$ (extension)

Axial Strain in AB:

- $\varepsilon_{AB} = \dfrac{l_{A'B'} - l_{AB}}{l_{AB}} = \dfrac{0.6 - 0.5}{0.5} = +0.2\ in/in$

General Definition of Axial (Extensional) Strain

Consider a line AB in an object as shown in the Figure 2.13 subjected to a combination of forces. After the application of forces the line elongates (or contracts) to A'B',

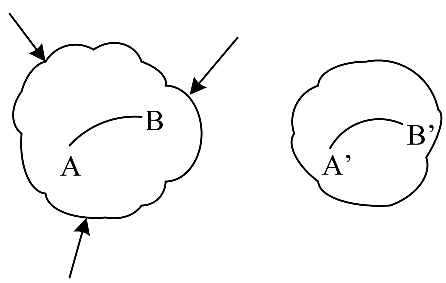


Figure 2.13: Object and line AB before and after deformation

Let the length of the line in the originally undeformed body be L and the length after deformation be L_1. If ε is the extensional strain at any point on the line then the incremental change (ΔL) of any infinitesimal length of the line (dL) can be used to define the new length of the infinitesimal segment (dL_1) as follows

$$dL_1 = dL + \Delta L$$
$$\Delta L = \varepsilon dL$$

Important: In the above equation we consider an infinitesimal segment since the extensional strain at any point on the line can have a *different* value. Consequently, the new length of the line can be expressed in an integral form as

$$L_1 = \int_A^B dL_1 = \int_A^B (1+\varepsilon)dL$$
$$L_1 = \int_A^B dL + \int_A^B \varepsilon dL$$
$$L_1 = L + \int_A^B \varepsilon dL$$

In general, ε can be any function, which can then be integrated.

Special Case: Axially Loaded Bar subjected to a constant axial force

Consider a bar having a constant cross sectional area subjected to a constant axial force through out as shown in Figure 2.14.

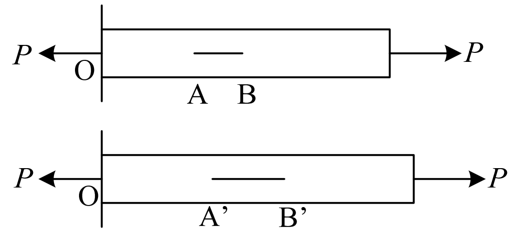


Figure 2.14: Axially loaded bar

Two situations are now considered to derive the expressions for change in length.

a) Small Strain Deformation: If the elongation in the bar is such that change in the total elongation is very small, it is reasonable to assume that the extensional strain value ε is *constant* throughout the bar. In this condition the change in length can be written as follows

$$\Delta L = \int \varepsilon dL = \varepsilon \int dL = \varepsilon L$$

or

$$\varepsilon = \frac{\Delta L}{L} = \frac{\text{change in length}}{\text{original length}}$$

b) Large Strain Deformation: However, if the elongation of the bar is such that the new length is significantly larger than the original length (say a one *inch* segment becomes 1.5 *inch*) then the strain equation defined by Equation (2.6) cannot be used. In this case it must be noted that reference length (L) keeps changing. Within this length L, consider an infinitesimally small length L_1 which changes to L_2. Hence the strain (called the true strain) can be expressed as

$$\varepsilon_{true} = \int_{L_1}^{L_2} \frac{dL}{L} = \ln(L)\big|_{L_1}^{L_2}$$

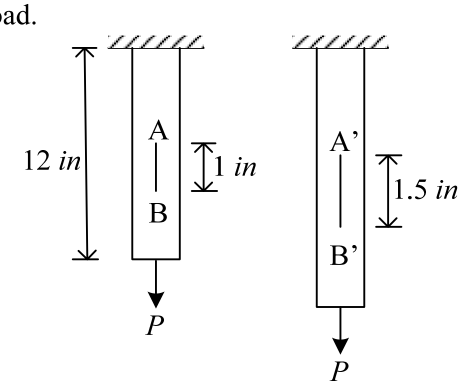
$$= \ln(L_2) - \ln(L_1) = \ln(\frac{L_2}{L_1}) = \ln(\frac{L_1 + \Delta L)}{L_1}) \qquad \text{Eqn (2.7)}$$

$$= \ln(1+\varepsilon)$$

In the equation (2.7) above the true strain (also called as log strain) value has been expressed in terms of the small strain. The problem of large strains is not addressed in this book and will be discussed in advanced courses such as Continuum Mechanics.

Small strains and True Strains

The conceptual example shown below gives an insight into how the strain definition in a large strain problem must be viewed. Consider a twelve *inch* rod, as shown in the figure below subjected to an axial force as shown. Assume that a gage length *AB* of one *inch* becomes 1.5 *inch* after the application of the load.



The computation of small strain ε_0 and the true strain is shown below.

$$\text{Small (Engineering) strain: } \varepsilon_o = \frac{\Delta l}{l_o} = \frac{(1.5-1.0)\ in}{1.0\ in} = 0.5$$

It can be seen that the gage length l_o used, the denominator, remains constant through out the entire deformation process (one inch). A strain of 0.5 or fifty percent is clearly an example of a rod undergoing large deformations.

In the computation of large strains the gage length is not constant. The computation is shown below where the problem is broken down into ten segments wherein the gage length is increased to the new value for every computation of the strain. It has to be kept in mind that the equation used to define and compute strain as the change in length divided by the gage length is valid only for small strains; hence the usage in the manner shown below is valid. In other words, the computation of strain (true strain) for the large deformation case is broken down into many small strain problems.

$$\text{Strain Definition: } \varepsilon_x = \frac{\Delta l}{l}$$

l	Δl	ε_x
1.0		
1.05	0.05	$\frac{0.05}{1} = 0.05$
1.1	0.05	$\frac{0.05}{1.05} = 0.0476$
1.15	0.05	$\frac{0.05}{1.1} = 0.04545$
1.2	0.05	$\frac{0.05}{1.15} = 0.0435$
1.25	0.05	$\frac{0.05}{1.2} = 0.0416$
1.3	0.05	$\frac{0.05}{1.25} = 0.04$
1.35	0.05	$\frac{0.05}{1.3} = 0.0385$
1.4	0.05	$\frac{0.05}{1.35} = 0.037$
1.45	0.05	$\frac{0.05}{1.4} = 0.0357$
1.5	0.05	$\frac{0.05}{1.45} = 0.0345$
		$\sum \varepsilon_x = 0.4132$

Table 3.1: True strain

True Stress-True Strain Curve: The mathematical equation representing true strain has been shown to be defined by a logarithmic equation. In a similar manner the true stress can also be shown be related to the engineering stress definition as shown below

$$\varepsilon_{true} = \ln(1 + \varepsilon_o)$$
$$\sigma_{true} = \ln(1 + \sigma_o)$$

The difference between the stress-strain graphs for the small deformation case $(\sigma_o - \varepsilon_o)$ and the large deformation case $(\sigma_{true} - \varepsilon_{true})$ is shown below. Note that in the large deformation case no portion of the stress strain graph goes into a descending portion.

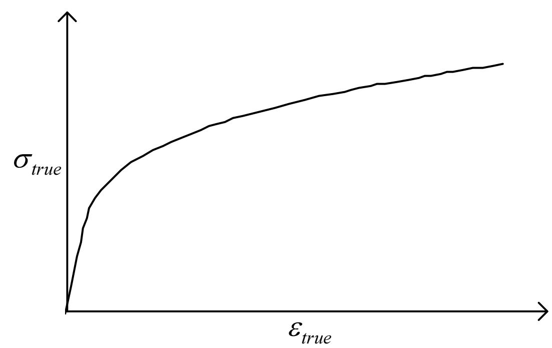
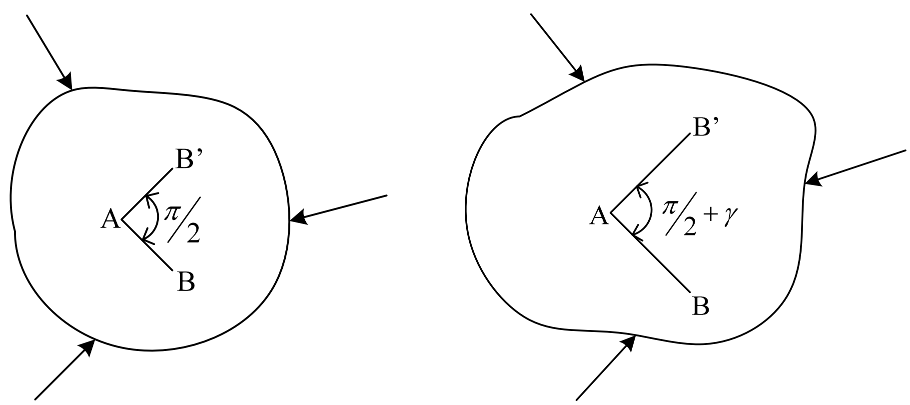


Figure 2.15: Graph of true stress vs. true strain

It can be seen that if the segments is increased to twenty or more the true strain value can be exactly obtained.

2.3.2 Shear Strain

The other basic type of strain, caused by shear stresses, is called shear strain. Unlike axial strain which measures the change in the length of a line element, the shear strain is a measure of the change in an angle.



(a) Before deformation (b) After deformation

Figure 2.16: Physical definition of shear strain

Consider an object subjected to forces as shown in Figure 2.16. Within this object consider two lines, AB and AB', which are perpendicular to each other. When the forces act on the object the angle between the two lines changes and let the new angle be ($\frac{\pi}{2} + \gamma$). The *change in the angle* γ is defined as the *shear strain***.**

- **Sign Convention:** +ve sign indicates that the angle between the two lines has increased and a –ve sign indicates a decrease in the angle.
- **Units:** *Radians* (all engineering angular units are measured in radians). '*Radians*' is a dimensionless quantity.

2.4 Hooke's Law Relating Stresses and Strains

In the chapter so far, we have discussed two primary physical quantities, namely stresses and strains. Stress is a measure of the intensity of the force while strain is a measure of the intensity of deformation. In order to have a tool to predict the deformation in the body subjected to a set of forces, we need a relationship which connects the stresses to strains. *Such a relationship cannot be derived and is determined purely by experiments, hence is called a law.*

Isotropic Material: An isotropic body is considered to be made of a material, which is assumed to be uniform in its material properties in all directions. Examples of isotropic materials are steel, brass, aluminum etc. All these materials can be visualized to be homogeneous and uniform.

This section describes some basic experiments that are performed in order to determine the relationship between stresses and strains. Two basic types of stresses and strains have been defined here relating

- Axial strains and axial stresses
- Shear strains and shear stresses

2.4.1 Young's Modulus and Poisson's Ratio

There are two basic and independent elastic constants that can be determined from a simple uniaxial test on a standard specimen. A uniaxial test represents a state of axial stress and axial strain in the specimen. Figure 2.17 shows a schematic representation of the uniaxial tension test.

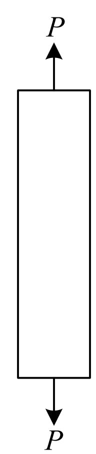


Figure 2.17: Schematic representation of a uniaxial tension test

No specific material (such as aluminum or steel) is considered in this explanation. The experimental observations typically are recorded as a force (P) and elongation (ΔL) relationship. Elongation is measured by using either a clip gage or a strain gage with a fixed and specified gage length (L). The axial stress and strain at any data point is then determined from the following equation.

$$\sigma = \frac{P}{A}; \quad \varepsilon = \frac{\Delta L}{L}$$

where, A is the area of cross section of the bar. Figure 18 shows a typical stress-strain curve for a tension test on a metallic specimen.

Hooke's Law: It can be seen from Figure 2.18 that in the region where the strains are low the experimental curve can be represented as a linear curve. This linear relationship, called Hooke's Law, can be expressed as

$$\sigma = E\varepsilon \qquad \text{Eqn (2.8)}$$

where, E is an elastic constant called the *Young's modulus* or *modulus of elasticity*. The slope of the line represents the modulus of elasticity and has the same units as that of stress. However it is typically three orders in magnitude to that of the stress value.

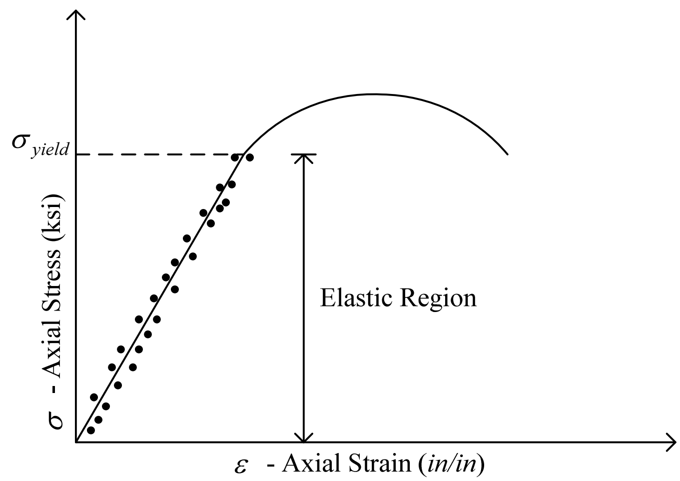


Figure 2.18: Results from a uniaxial tension test

Another observation that is recorded in a tension test corresponds to the observation that the cross sectional area at the middle of the specimen decreases as the elongation in the longitudinal direction occurs. This phenomenon, known as the *Poisson's effect*, represents the decrease in the lateral dimension due to a longitudinal elongation. Figure 2.19 shows a schematic representation of the axial strain and Poisson's effect.

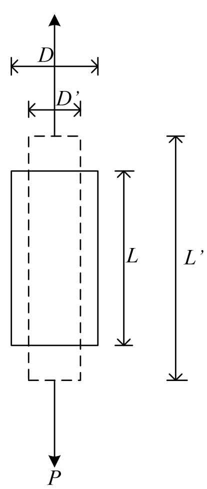


Figure 2.19: The axial strain and Poisson's effect

The strain in the longitudinal ($\varepsilon_{longitudinal}$) and the lateral ($\varepsilon_{lateral}$) can be defined as follows

$$\varepsilon_{longitudinal} = \frac{\Delta L}{L} = \frac{L' - L}{L}$$

$$\varepsilon_{lateral} = \frac{D' - D}{D}$$

In a typical tension test the longitudinal strain is positive (elongation) while the lateral strain is negative (contraction). Poisson's effect is now defined using a physical or elastic constant called *Poisson's Ratio (ν)* (Greek letter – pronounced as 'nu' or 'new').

$$\nu = -\frac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}}$$

The negative sign is introduced in order to have a positive constant value. For most metals the value of Poisson's Ratio ranges between 0.25 and 035. It will be shown in a later section that the value of this ratio can never exceed 0.5.

Table 2.2 gives typical values of the Elastic Modulus and Poisson's ratio for some commonly used materials in engineering.

Material	Young's Modulus		Poisson's Ratio
	ksi	*GPa*	
Aluminum	10,000	70	0.33
Brass	14-16,000	96-110	0.34
Concrete (compression)	2.500-4,500	18-30	0.1-0.2
Steel	29,000	210	0.28-0.3

Table 2.2: Table of material properties of some common materials

Material behavior and material properties are described in greater detail in a later chapter.

2.4.2 Shear Modulus

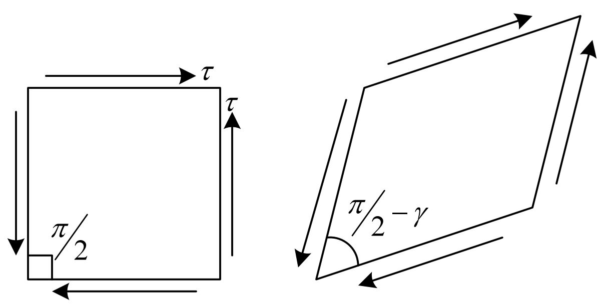


Figure 2.20: Shear stress-strain

Consider a cube subjected to a shearing action (shear stress τ) as shown in Figure 2.20, which shows the two dimensional elevation of the cube. Due to the action of the forces the bottom left right angle changes and this change in the angle is called the *shear strain* (γ). Considering the cube to be made of a metal, as the shear stress is increased it can be observed that the shear strain also changes and the relationship between the two is a linear one. This linear relationship can be expressed as

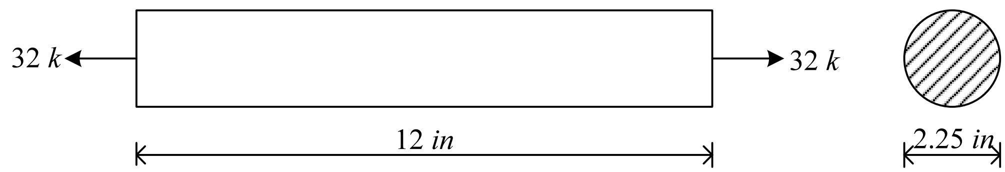
$$\tau = G\gamma$$

where, G is called the *Shear Modulus* or *Modulus of Rigidity,* and is the third elastic constant defined so far in this chapter. For any material, the elastic behavior can be *completely defined* by only *two elastic constants.* It will be shown in a subsequent chapter that the relationship between the shear modulus, elastic modulus and Poisson's ratio can be expressed as

$$G = \frac{E}{2(1+\nu)}$$

Example 2.4: Determination of elastic constants

Problem Statement: A circular rod made of aluminum is 2.25 *inches* in diameter and 12 *inches* in length is subjected to an axial force of 32 *kips.* At the end of the tension test the rod length was measured to be 12.00938 *inches* and its diameter at the center decreased to 2.249415 *inches*. Determine the two elastic constants for aluminum, namely Young's modulus E and the Poisson's ratio ν.



Required: Find the Young's modulus and Poisson's ratio of aluminum

Solution:

Step 1: Determine the change in length and diameter of the bar

$$\Delta l = 12.00938\ in - 12\ in = 0.00938\ in$$

$$\Delta d = 2.249415\ in - 2.25\ in = -0.000585\ in$$

Step 2: Determine the axial and transverse strain

$$\varepsilon_x = \frac{\Delta l}{l} = \frac{0.00938\ in}{12\ in} = 0.000782\ in/in.$$

$$\varepsilon_{transverse} = \frac{\Delta d}{d} = \frac{-0.000585\ in}{2.25\ in} = -0.00026\ in/in.$$

Step 3: Since this is a uniaxial test the equation of Poisson's ratio $\nu = -\frac{\varepsilon_{transverse}}{\varepsilon_{longitudinal}}$ can be used.

The elastic modulus and the Poisson's ratio is now determined

$$\nu = -\frac{\varepsilon_{transverse}}{\varepsilon_{longitudinal}} = -\frac{\varepsilon_{transverse}}{\varepsilon_x} = -\frac{-0.00026}{0.000782} = 0.333$$

$$E = \frac{\sigma_x}{\varepsilon_x}$$

$$\sigma_x = \frac{P}{A} = \frac{32\ kips}{\pi(1.125)^2} = 8,030\ psi$$

$$E = \frac{\sigma_x}{\varepsilon_x} = \frac{8,030\ psi}{0.000782} = 10,268\ psi$$

Hence

$E = 10,300\ ksi$ and $\nu = 0.333$

2.5 Design Issues

The goal of the study of mechanics of materials is to establish the fundamental principles based on which any structure such as buildings, bridges, cars, micro devices etc. can be designed. It will be eventually seen that actual designs, while based on principles of mechanics of materials, incorporate many practical conditions and issues.

Material	Yeild Stress	
	ksi	*MPa*
Aluminum	2.9	20
Brass	10.15 – 79.77	70-550
Iron (Wrought)	30.46	210
Steel	40.6 - 232	280-1600

Table 2.3: Yield stress values for some common materials

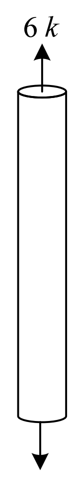
The design of a structure involves the determination of the cross section geometry necessary to withstand the loads applied to it. The uniaxial test on materials is a very popular method to determine the yield stress f_y of any material. For isotropic materials this is statistically a constant value. The design process however uses the *allowable stress* values which is the yield stress value divided by a factor of safety *(FOS).* The factor of safety accounts for uncertainties in material properties and actual loads. Table 2.3 gives a partial list of some commonly used materials and their yield stress values. The factors of safety can also vary with the type of behavior that the structure is experiencing.

The basic design procedure for axially loaded members or bolts in shear or in bearing is to first find the force acting on the member P and then divide the force by the allowable stress value f_{all} in order to find the area as shown below

$$A_{req.} = \frac{P}{f_{all.}}$$

From the area required either the bolt diameter or the thickness and the width of the plate required are determined.

Consider a simple example of a circular rod, as shown in the figure, made of steel and is required to carry an axial force of 6 *kips.* Given that the yield stress of steel is 50 *ksi* and using a factor of safety of 1.5, it is required to find the diameter of the steel rod.



$$f_{all.} = \frac{f_y}{FOS} = \frac{50\ ksi}{1.5} = 33.33\ ksi$$

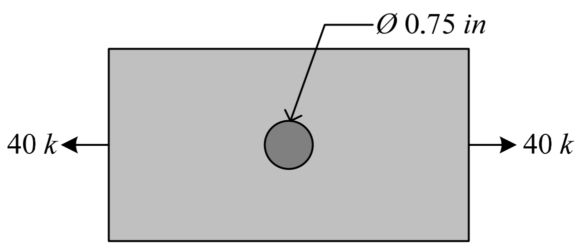
$$A_{req.} = \frac{P}{f_{all.}} = \frac{6\ k}{33.33\ ksi} = 0.18\ in^2 = \frac{\pi d^2}{4}$$

$$d \geq 0.478\ in$$

$$\therefore \text{Provide } d = 0.5\ in.$$

Gross and Net Areas: In tension members the concept of gross and net areas is an important entity. The section taken across the net area is the least area available for the material to resist the force; hence it experiences the greatest stress value. Consequently, the allowable stress value is first reached in the net section. Hence, the designer must exercise care in using the net area in order to determine the width or the thickness of the section.

Consider the plate with a hole subjected to tensile forces, as shown in the figure. If the thickness of the plate to be designed is ½ *in* determine the width of the plate. The plate is made of steel as in the previous problem and is subjected to a force of 40 *kips.*



$$A_{net} \geq \frac{P}{f_{all.}} = \frac{40k}{33.33\ ksi} = 1.2\ in^2$$

$$A_{net} = A_g - d_{hole} t_p$$

$$d_{hole} = 0.75\ in.$$

$$1.2 = bt_p - d_{hole} t_p$$

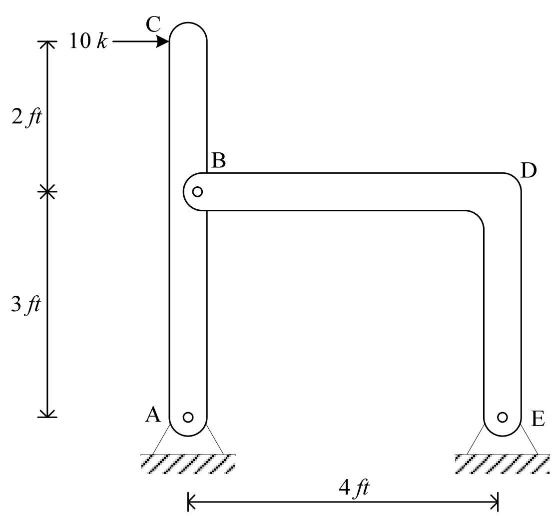
$$1.2 = b(0.5) - (0.75 \times 0.5)$$

$$b \geq 3.15\ in$$

$$\text{Provide } b = 3.25\ in$$

Example 2.5: Design problem

Problem Statement: If the frame in the problem is made of members that are 3/4 *in*. thick, if the allowable stresses are $\tau_{all} = 24\ ksi$ and $f_{bearing} = 33.33\ ksi$, determine the diameter of the bolt required.

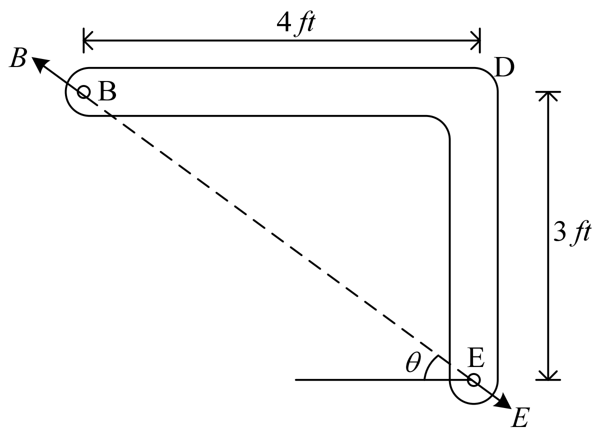


Required: Find the diameter of the bolt

Solution:

Step 1: Determine the force acting at C. For this a frame analysis is required. From the figure it can be seen that member ABC is a multi-force member while member CDE is a 2-force member.

Equilibrium of a 2F member: Forces are acting at C and E only. Hence, the resultant force goes along the line joining C and E. From geometry, the angle that the resultant is shown in the figure.



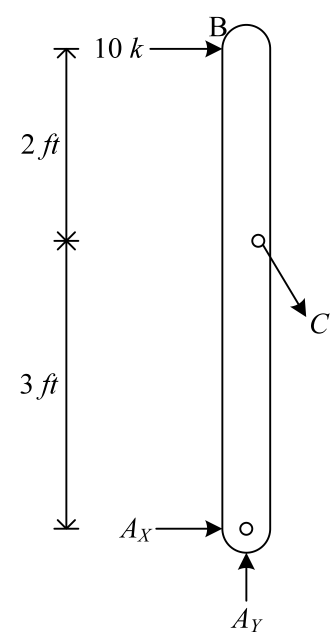
$$\tan\theta = \frac{3}{4}$$

$$\theta = 36.87^o$$

Step 2: Draw the free body diagram of the multi force member ABC, determine the force at C. The force at C is the force acting in the bolt.

$$-B\sin\theta(3) - 10(5\,ft) = 0$$

$$B = -20.83\ kips$$



Step 3: The force in the bolt is 20.83 *kips*. The design of the bolt diameter is to be determined from both the shear stress and the bearing stress criterion.

From the shear stress criterion:

$$A_{bolt} = \frac{B}{\tau_{all.}} = \frac{20.83\ kips}{24\ ksi} = 0.87\ in^2$$

$$0.87\ in^2 = \frac{\pi d^2}{4}$$

$$d \geq 1.1\ in$$

$\therefore$ Provide $d = 1\,{}^{1}\!/_{8}\ in.$

From the bearing stress criterion:

$$A_b = \frac{B}{f_{all.}} = \frac{20.83\ kips}{33\ ksi} = 0.625\ in^2$$

$$A_b = d_{bolt} t_{plate}$$

$$d_{bolt} \geq \frac{0.625}{0.75}$$

$$d \geq 0.8332\ in$$

Provide $d = 1\,{}^{1}\!/_{8}\ in$

It can be seen that shearing stress criterion gives a greater diameter, which is the diameter to be provided.

2.6 Summary

The key terms introduced in this chapter are summarized below.

1. **Tension (Normal Stress):** This type of stress is caused by force acting at the centroid of the section, in a direction perpendicular and away from the area. The effect on the body is to stretch it.
2. **Compression (Normal Stress):** This type of stress is caused by a force acting at the centroid of the section, in a direction perpendicular and into from the area. The effect on the body is to crush it.
3. **Single Shear:** A shear stress acts parallel to an area. A pin is said to be in single shear if only one cross section area of the bolt is effective in resisting the force.
4. **Double Shear:** A pin or bolt is said to be in double shear if two areas of the bolt are effective in resisting the applied force as seen in Figure 2.5.
5. **Bearing Stress (Normal Stress):** A bearing stress is compressive nature. However, this is a stress on the surface between two objects. The characteristic of this stress that it acts perpendicular to the surface and into the surface.
6. **Displacement:** A specific point is said to displace when the object deforms. The displacement term is used to denote the actual movement of a *point*.
7. **Elongation** - of a *region*; This measure the stretch of a region in the bar
8. **Axial Strain** - in the *region*; The axial strain in a region is defined as the elongation of region divided by its original length
9. **Shear Strain:** When the forces act on the object the angle between the two lines changes. The *change in the angle* γ is defined as the *shear strain* and is measure in radians.
10. **Elastic or Material Properties:** The elastic properties of a material are its Young's modulus, Poisson's ratio, Shear Modulus and the Bulk Modulus,
11. **Isotropic Material:** A homogeneous material with a uniform form is said to isotropic if it has identical material properties in all directions.
12. **Young's Modulus:** The Young's modulus or the Elastic modulus (E) is a measure of the initial stiffness of the material and is determined from the slope of the stress strain graph from a tension test of the material. It has the units of *GPa* or *ksi.*
13. **Poisson's Ratio:** The Poisson's ratio quantifies the Poisson's effect which is the phenomenon of the transverse diameter shortening upon being pulled in the longitudinal direction. It has no units.
14. **Shear Modulus:** The shear modulus (G) relates the shear strain to the shear stress and represents the material's ability to resist a shear strain.

2.7 Problems:

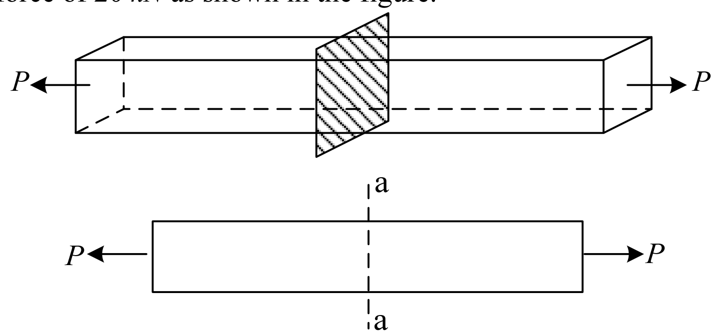
2.1 Define axial direction.

2.2 Define a normal stress.

2.3 A steel bar has a radius of 10 *mm* and is axially loaded by force of 4 *kN*. Determine the axial stress and express your answer in *MPa* units.

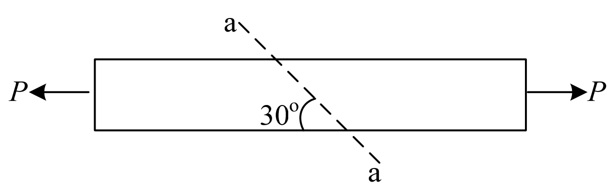
2.4 An aluminum rod of 0.5 *in* diameter is experiencing an axial stress of 30 *ksi*. What is the axial force in the bar?

2.5 Determine the axial stress in a rectangular bar of cross section area 10*mm* x 50*mm* and loaded by a force of 20 *kN* as shown in the figure.

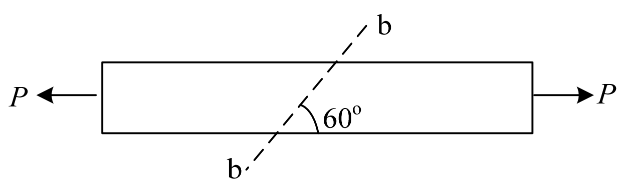


2.6 Determine the axial stress and shear stress on the area along section a-a and b-b. The rectangular bar is having the cross section area and loading same as in problem 2.5.

a.

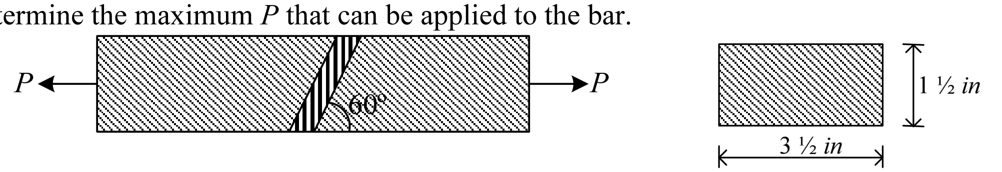


b.



2.7 The figure shown in problem 2.6, if the shear stress on section a-a is 350 *MPa*, determine:
a) Axial force in the rod.
b) Axial stress on the section b-b

2.8 Two pieces are glued as shown in the figure, given that the cross section of the wood at the ends is 1½ *in* x 3½ *in* and if the maximum shearing stress that the glue can take is 3 *psi* determine the maximum *P* that can be applied to the bar.

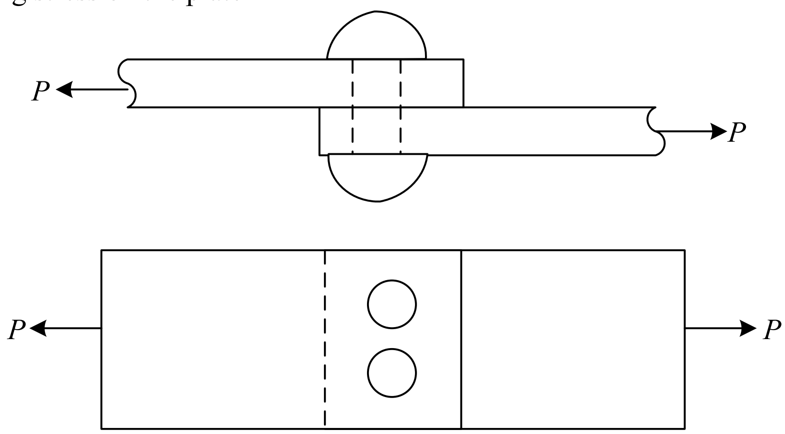


2.9 In the above example if the maximum axial stress in the wood is 5 *psi* determine the maximum *P* that can be applied to the bar.

2.10 Using the data and information from problem 2.8 and 2.9, if the maximum tensile stress in the glue is 5 *psi*, determine the maximum load *P* that can be applied.

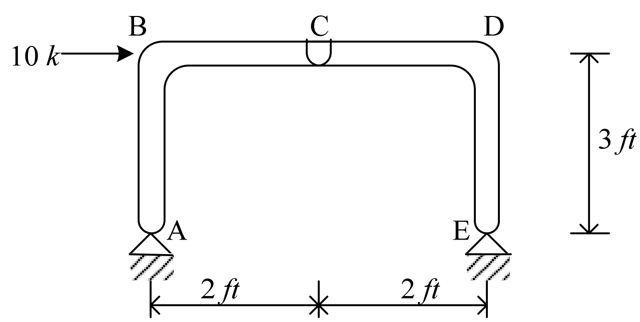
2.11 Two plates are connected as shown in the figure. If the width of the plates is 3 *in*, the thickness is ½ *in* and the diameter of the bolt is ¾ *in* determine

a) maximum tensile stress in the plate
b) shear stress in the bolt
c) bearing stress on the plate



2.12 In problem 13, if the maximum tensile in the plate and the maximum shear stress in the bolt is 6 *ksi* and 25 *ksi*, determine the maximum load *P* that can be applied to the plate.

2.13 The frame in the figure is loaded as shown. If the bolt at C has allowable shear stress of 20 *ksi*, determine the required diameter of the bolt. Assume that the bolt is in single shear.



2.14 A circular rod made of aluminum is 2 *inches* in diameter and 10 *inches* in length is subjected to an axial force of 25 *kips.* At the end of the tension test the rod length was measured to be 10.0025 *inches* and its diameter at the center decreased to 1.9415 *inches*. Determine the elastic constants for aluminum, namely Young's modulus *E,* the Poisson's ratio ν and Shear Modulus *G.*