Conceptual Dynamics

An Interactive Text and Workbook

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PART II: KINEMATICS

CHAPTER 4: KINEMATICS OF RIGID BODIES

CHAPTER OUTLINE

CHAPTER SUMMARY

In this chapter we begin to analyze the motion of rigid bodies. Recall that rigid bodies have size as well as mass. Therefore, the rotation of a rigid body must be considered in addition to its translation. We will begin to look, not only at a body's linear motion, but also at its angular motion characteristics (e.g. angular velocity and angular acceleration). We will start simple with pure rotation and then move on to general planar motion.

4.1) RIGID-BODY MOTION

4.1.1) WHAT IS A RIGID BODY?

There are certain circumstances where a particle representation of a body in motion is not sufficiently accurate. A clear indication that a particle is not suited for the job is if rotation is an important part of the body's motion. In this case and in other circumstances, we will need to represent the object under consideration as a rigid body. A **rigid body** is different from a particle because it has both mass and size, whereas, a particle only has mass. Therefore, with a rigid-body model, the body's rotation must be considered in addition to its translation.

4.1.2) HOW DOES A RIGID BODY MOVE?

A rigid body can move in pure translation, pure rotation, or general planar motion. **Pure translation** occurs when the body goes from one location to another without changing its orientation. All points on the rigid body undergoing pure translation have the same velocity and acceleration (i.e. $\mathbf{v}_A = \mathbf{v}_B$, $\mathbf{a}_A = \mathbf{a}_B$). This is because the paths taken by each of the points are parallel to one another. **Pure rotation** occurs when the body is rotating about some fixed axis and only the orientation of the body is changing. In this case, all parts of the body move in circular paths about the fixed axis. All points on the body rotate through the same angular displacement at the same time. If the body is undergoing **general planar motion**, it is simultaneously translating and rotating.

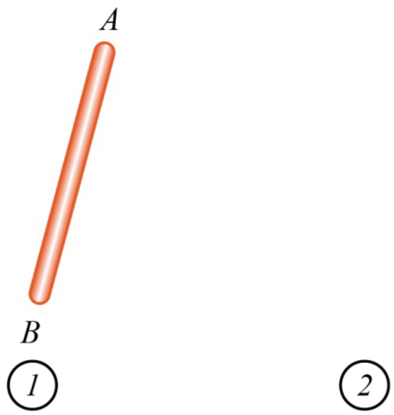
Our prior study of translational kinematics involved the investigation of the relationship between the displacement, velocity, and acceleration of a body. Completely analogous relationships exist for the rotational kinematics of a rigid body and are shown below. Since a rigid body can both translate and rotate, both sets of equations apply, given a certain set of circumstances. We will look at these sets of equations in detail in the upcoming sections. Specifically, we will investigate the relationship between a body's angular displacement θ, its angular velocity ω, and its angular acceleration α.

Rotational Equations	**Translational Equations**
$\omega = \dfrac{d\theta}{dt}$	$v = \dfrac{ds}{dt}$
$\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$	$a = \dfrac{dv}{dt} = \dfrac{d^2 s}{dt^2}$
$\alpha \; d\theta = \omega \; d\omega$	$a \; ds = v \; dv$

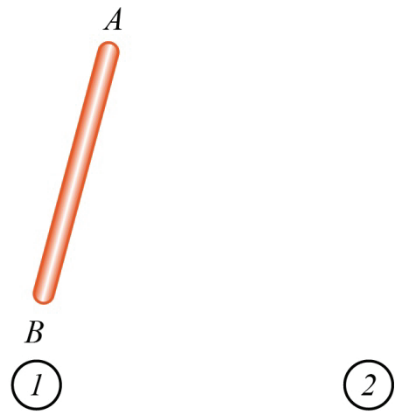
What is the difference between the motion of a particle and the motion of a rigid body?

Conceptual Example 4.1-1

Draw possible paths for the motion of bar AB as it moves from point 1 to point 2 in pure translation.

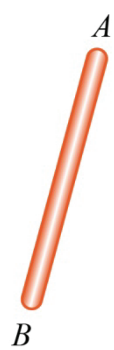


Rectilinear motion



Curvilinear motion

Draw the path of motion for bar AB as it rotates about point B in pure rotation



4.2) PURE ROTATION

4.2.1) ROTATION ABOUT A FIXED AXIS

We will first consider the case of a rigid body rotating about a fixed axis as shown in Figure 4.2-1. **Pure rotation** occurs when a body rotates about a fixed non-moving axis. Under the rigid-body assumption, the distance between any two points on the body remains constant. Therefore, each point on the body can be considered to be moving in concentric circles about the fixed axis. Fixed-point O, shown in Figure 4.2-1, is a point the coincides with the fixed axis, in this case, the z-axis.

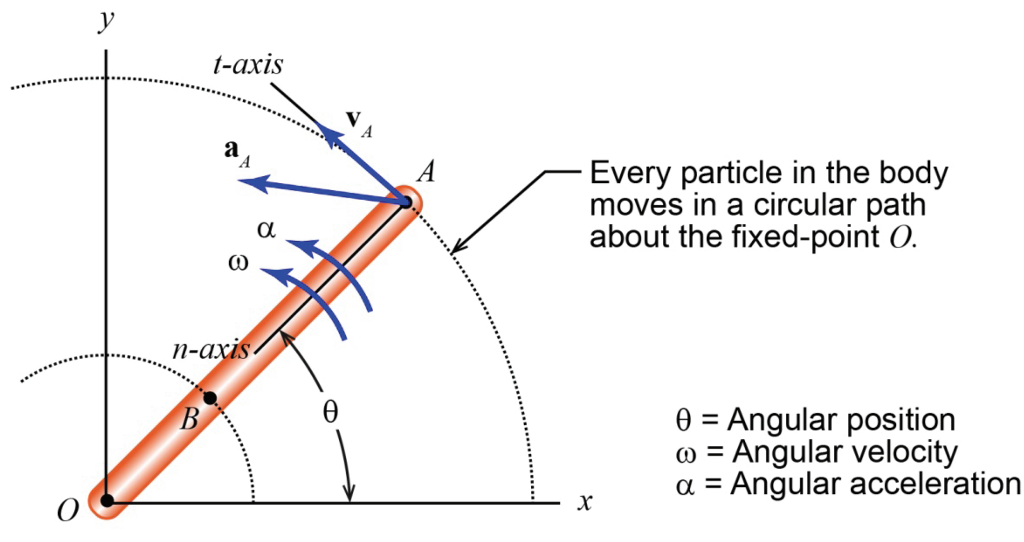


Figure 4.2-1: Fixed-axis rotation

4.2.2) ANGULAR KINEMATIC RELATIONSHIPS

In previous chapters, we have studied linear position (s), linear velocity (v) and linear acceleration (a). Each of these motion parameters are related to each other through kinematic relationships (i.e., $dv = ds/dt$, $a = dv/dt$, $ads = vdv$). In this section, we will learn about the parameters that describe the rotational characteristics of a body's motion. These motion parameters are angular position (θ), angular velocity (ω) and angular acceleration (α). As one would expect, angular velocity is the time derivative of angular displacement (i.e. $\omega = \dot{\theta}$), and angular acceleration is the time derivative of angular velocity (i.e. $\alpha = \dot{\omega}$) as shown in Equations 4.2-1 through 4.2-3. In general, units of radians, rad/sec and rad/sec^2 are employed for angular displacement, angular velocity and angular

Units of Angular Position (angle)
SI derived and accepted Units:
- radians (rad)
- degrees ($^{\circ}$)

Units of Angular Velocity (angle per time)
SI derived Units:
- radians per second (rad/s)

US Customary Units:
- revolutions per minute (rpm)

Units of Angular Acceleration (angle per time squared)
SI derived Units:
- radians per second squared (rad/s^2)

acceleration, respectively. However, degrees and revolutions per minute (rpm) are also common units for angular position and angular velocity, respectively. It is always a good idea to convert non-radian units into radians before performing calculations to avoid errors due to unit mismatch.

Angular speed: $$\boxed{\omega = \frac{d\theta}{dt}} \quad (4.2\text{-}1)$$

Angular acceleration: $$\boxed{\alpha = \frac{d\omega}{dt}} \quad (4.2\text{-}2)$$

$$\boxed{\alpha\, d\theta = \omega\, d\omega} \quad (4.2\text{-}3)$$

θ = Angular displacement (rad)
ω = angular speed (rad/s)
α = angular acceleration (rad/s^2)
t = time

Even though Equations 4.2-1 through 4.2-3 are not written as such, angular displacement, angular velocity and angular acceleration are vector quantities. Specifically, all of the angular motion parameters have a magnitude and direction of rotation. A simplified way of looking at this is that the rotational direction can be clockwise or counterclockwise. This clockwise/counterclockwise direction can be expressed as a negative or positive sign attached to the magnitude. For more complex systems, the rotational direction is expressed with unit direction vectors, usually $\mathbf{k}$ for a planar motion (i.e. 2-D) system, or as a combination of unit direction vectors for motion in three dimensions. Whether you indicate the direction as being clockwise or counterclockwise, or with unit direction vectors such as $\mathbf{k}$, these quantities have both magnitude and direction making them vectors.

An easy way to determine the direction of angular motion and whether it is positive or negative is through the use of the right-hand rule. Here's how it works: take the fingers on your right hand and curl them in the direction of rotation. The direction of the vector representing the angular motion is indicated by the direction of your thumb. For example, if your thumb is pointing along the positive y-axis, the direction is $\mathbf{j}$. If your thumb is pointing along the negative x-axis it is $-\mathbf{i}$.

For planar problems, clockwise motion corresponds to a vector pointing in to the paper and counterclockwise motion corresponds to a vector pointing out of the paper (as in Figure 4.2-1). These vectors can be expressed using the unit vector $\mathbf{k}$ that is positive out of the paper and negative into the paper. For general three-dimensional motion, the direction of rotation can be in any direction and must be expressed as a combination of components in the $\mathbf{i}$-, $\mathbf{j}$- and $\mathbf{k}$-directions.

4.2.3) VELOCITY AND ACCELERATION OF A POINT

Remember the games that you used to play outside as a kid, such as jumping rope or skip ball, see Figure 4.2-2. Think about how the jump rope and skip ball move when you start playing. Is the velocity along the jump rope the same at every point? Is the velocity of the ball at the end of the skip ball rope the same as the ring around your ankle? The answer to both of these questions is no. We can idealize these games and assume that the velocity at the hands (in the case of jumping rope) and the ankle (in the case of skip ball) is very near zero. This makes these situations similar to fixed-axis rotation. As we move out from the hands or ankle, the velocity will increase until we reach a maximum velocity at the apex of the jump rope or at the ball. The key point that these two examples illustrate is that the velocity of every point on a rigid body undergoing fixed-axis rotation may be different. The velocity of a particular point depends on the distance it is away from the fixed axis. The magnitude of the velocity increases the further away you get from the fixed axis of rotation.

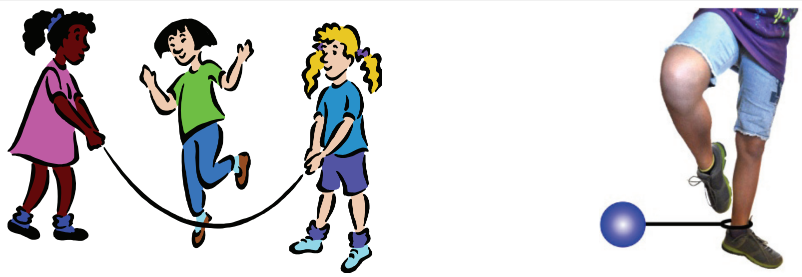


Figure 4.2-2: Jump rope and skip ball

The velocity of each point on a rigid body can be derived by considering the bar, shown in Figure 4.2-5, that moves from position *1* to *2*. The arc length scribed by point A is given by the arc length equation $ds = rd\theta$. Taking the time derivative of the arc length equation gives us an equation for the velocity of point A, which is $v_A = r_A\dot{\theta}$. This equation shows the relationship between velocity and distance from the fixed axis as previously mentioned. This equation may be extended to any point on the body. We also know that the velocity is always in the tangential direction. The velocity of any point on a rigid body, in pure rotation, is given by Equation 4.2-4.

The acceleration of each point on a rigid body can be obtained by using the normal and tangential coordinate acceleration equation presented in the chapter on *kinematics of particles - curvilinear motion* and making the substitution for the velocity with the one given in Equation 4.2-4. The normal and tangential acceleration equation is $\mathbf{a} = \dot{v}\,\mathbf{e}_t + (v^2/\rho)\mathbf{e}_n$. Substituting $v = r\omega$ and $\dot{v} = r\dot{\omega} = r\alpha$, we get the Equation 4.2-5. This equation may be used to calculate the acceleration of any point on a rigid body undergoing pure rotation. This equation can also be derived from the expression for acceleration in the polar coordinate frame when r is constant.

Velocity: $$\boxed{\mathbf{v}_A = r_A\omega\mathbf{e}_t = r_A\omega\mathbf{e}_\theta} \quad (4.2\text{-}4)$$

Acceleration: $$\boxed{\mathbf{a}_A = r_A\alpha\,\mathbf{e}_t + r_A\omega^2\mathbf{e}_n = r_A\alpha\,\mathbf{e}_\theta - r_A\omega^2\mathbf{e}_r} \quad (4.2\text{-}5)$$

$\mathbf{v}_A$ = linear velocity of point A
$\mathbf{a}_A$ = linear acceleration of point A
r_A = distance of point A from the fixed axis
ω = angular speed
α = angular acceleration

Note that while velocity and acceleration on the rigid body are dependent on radial location (i.e. distance away from the fixed axis of rotation), the body as a whole has only a single angular velocity ω and angular acceleration α. Consider for example the body shown in Figure 4.2-1. A line drawn from point O to point A will sweep out the same angle as a line drawn from point O to point B for the same time interval. The distance travelled by point A, however, will be significantly larger than the distance travelled by point B during that same time interval. This is because point A is located at a radius that is farther from point O than it is from point B.

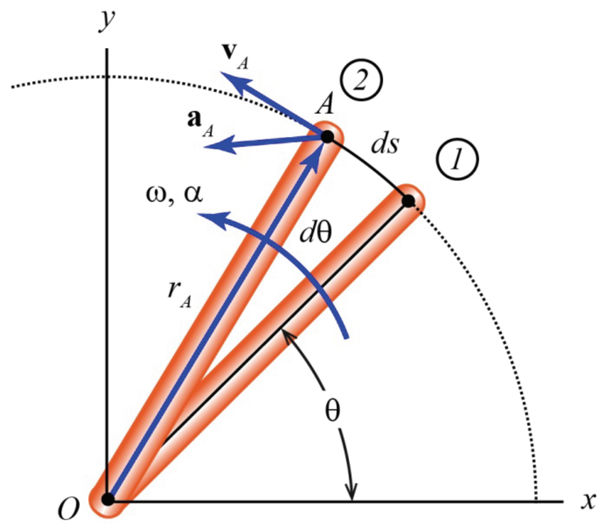


Figure 4.2-3: Velocity of a body in pure rotation

Conceptual Example 4.2-1

What is the velocity direction of any point on a rigid body undergoing planar pure rotation? Choose all that are true.

a) Tangent to its path of motion.
b) Perpendicular to the line drawn from the fixed axis to the point in question.
c) $\mathbf{e}_t$
d) $\mathbf{e}_\theta$
e) None of the above.

Example 4.2-2

Use the arc length equation to derive the velocity of point A on the rigid body shown in Figure 4.2-3 which is undergoing pure rotation.

Example 4.2-3

Derive the acceleration of any point on a rigid body undergoing pure rotation in terms of the radial distance (r), angular speed (ω), and angular acceleration (α) starting with the two equations given below.

$$\mathbf{a} = \dot{v}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n \qquad \mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

Equation 4.2-4 and 4.2-5 gives the velocity and acceleration for an arbitrary point on a rigid body. When you use these equations, you are forced to work in the normal and tangential or the polar coordinate systems. If you wished to express the velocity and acceleration of a particle as a vector in rectangular coordinates, geometry could be employed to decompose given vectors into their rectangular components, but this can be very tedious. Another approach for finding the vector expressions for velocity and acceleration in any non-rotating coordinate system is to use the following relationships involving the cross product. For instructions on how to perform a cross product, refer to Appendix B.

Velocity: $$\boxed{\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_{A/O}} \quad (4.2\text{-}6)$$

Acceleration for 2-D applications: $$\boxed{\mathbf{a}_A = \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}} \quad (4.2\text{-}7)$$

Acceleration for 3-D applications: $$\boxed{\mathbf{a}_A = \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})} \quad (4.2\text{-}8)$$

$\mathbf{v}_A$ = linear velocity of point A
$\mathbf{a}_A$ = linear acceleration of point A
ω = angular velocity
α = angular acceleration
$\mathbf{r}_{A/O}$ = position of point A relative to the fixed axis O

Conceptual Example 4.2-4

Consider the figure of the wind turbine shown.

1) What is the velocity of the center O?

2) The angular velocity of point A (a point midway between the center O and the tip of the blade) is __________ that of point B (a point at the tip of the blade).

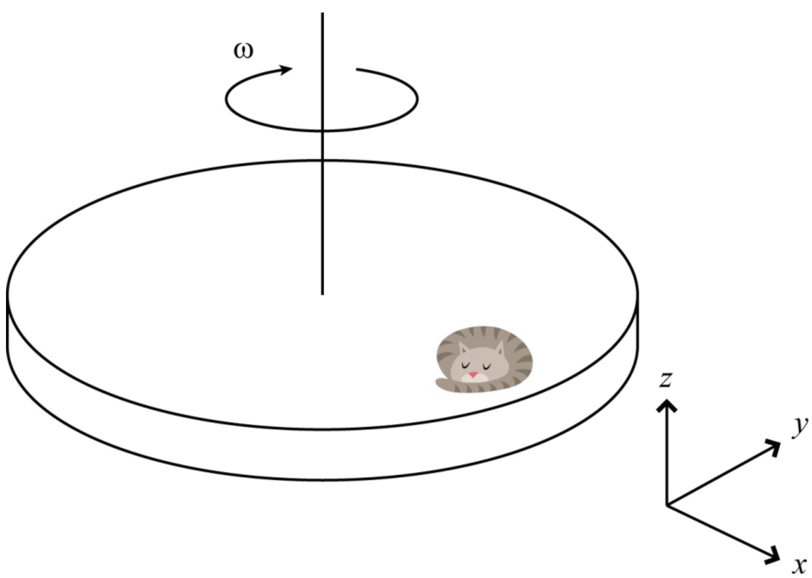
   a) half
   b) the same as
   c) twice
   d) None of the above.

3) The velocity of point A is __________ that of point B.

   a) half
   b) the same as
   c) twice
   d) None of the above.

Conceptual Example 4.2-5

A cat sits without slipping on the edge of a spinning disk. The disk is turning in the direction shown and slowing down.

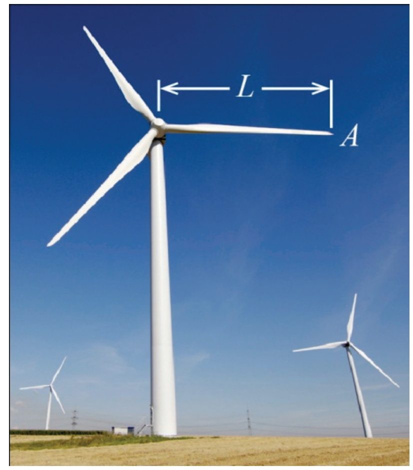


1) At the instant shown, what is the direction of the radial component of the cat's acceleration?
2) At the instant shown, what is the direction of the tangential component of the cat's acceleration?
3) At the instant shown, what is the direction of the angular velocity of the cat?

   a) $+x$ b) $-x$ c) $+y$ d) $-y$ e) $+z$ f) $-z$

Example Problem 4.2-6

The blades on the wind turbine are turning with an angular speed of ω_o = 30 rpm in the clockwise direction. If the blade length is L = 120 ft and they are given a constant angular acceleration of $\alpha = 0.2$ rad/s^2 in the clockwise direction, determine the angular velocity and the magnitude of the acceleration of point A on the tip of the blade when t = 1 minute.



Given:

Find:

Solution:

4.2.4) GEARS AND BELTS

Shafts are a common rotational motion component in machines. Often it is necessary to transfer the rotation of one shaft to another. Transferring the motion of one shaft to another on a one-to-one basis is not always useful. This means that if the first shaft is rotating with an angular speed of ω_1, the second shaft will rotate with the same angular speed. Many times we want to transfer motion and change the rate at which the shaft is rotating. Take, for example, the accessory components in your car (e.g. compressor, water pump, alternator). These accessory components are driven by the crankshaft, which in turn, is driven by the piston motion. The crankshaft rotates at one speed and the other components that run off of it need to run at different speeds. Therefore, the crankshaft rotation needs to be transferred and the speed needs to be modified. This is accomplished by using a belt (i.e. the serpentine belt) and a series of pulleys (see Figure 4.2-4). It is easy to see how the belt transfers the rotational motion, but how does the speed change if the belt is continuously running at a constant speed? This is accomplished by using different sized pulleys.

In any belt-pulley system, if you decrease the size of the accessory pulley relative to the drive pulley, that component will run faster than the driving shaft. On the other hand, if you increase the size of the accessory pulley relative to the drive pulley, the component will run slower than the driving shaft. Gears and contacting wheels can also be used to accomplish similar results.

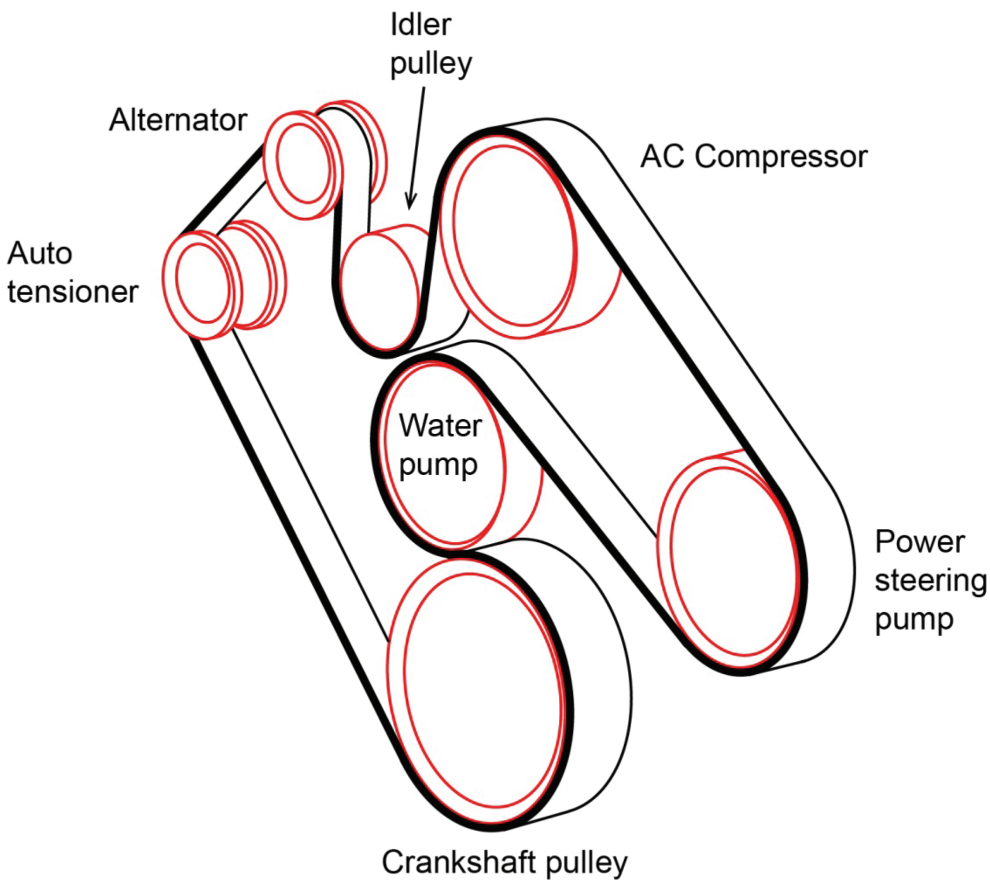


Figure 4.2-4: Accessory drive belt and pulley system in an automobile

As discussed above, gears and pulleys can be employed to change the speed at which a body is rotating. Consider one gear driving another gear as shown in Figure 4.2-5. The driven gear is given an angular speed of ω_1 and will turn the other gear with an angular speed of ω_2. The two angular speeds are related through the ratio of their radii. This follows from the fact that the linear velocity of the contact points for each of the gears are equal; this is assuming that the gears do not slip with respect to one another.

$$v_{1,\text{point of contact}} = v_{2,\text{point of contact}}$$
$$r_1\omega_1 = r_2\omega_2$$
$$\omega_2 = \frac{r_1}{r_2}\omega_1$$

The above relationship works for gears, pulleys with belts, and non-slipping wheels in contact. The logic for a pulley-belt system follows from the fact that the belt has the same speed at every point. Which means that every point on the rim of a pulley in contact with the belt will have the same speed. The speed shared by the belt and pulley is proportional to the radius of the pulley (i.e. $v = r\omega$). Gears have an additional relationship shown below. Their angular speeds can be related through the ratio of their number of teeth (N). This is because the gear radius is proportional to how many teeth it has.

$$\omega_2 = \frac{N_1}{N_2}\omega_1$$

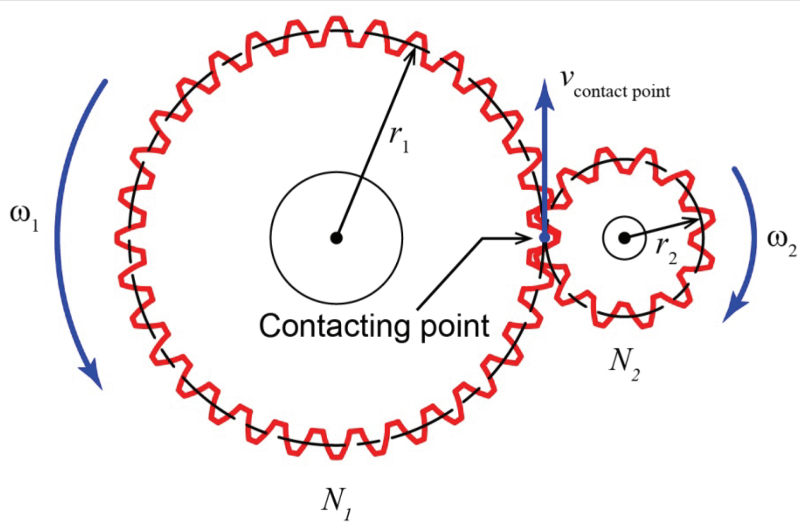


Figure 4.2-5: Two gears in contact

A similar relationship for the angular acceleration may be derived. The tangential acceleration ($a_t = r\alpha$) of the gear's contacting point or the belt/pulley interface must be shared and equal. Therefore, the following relationship for the angular acceleration between two gears or a belt and two pulleys are as follows.

$$a_{t1,\text{point of contact}} = a_{t2,\text{point of contact}}$$

$$r_1\alpha_1 = r_2\alpha_2$$

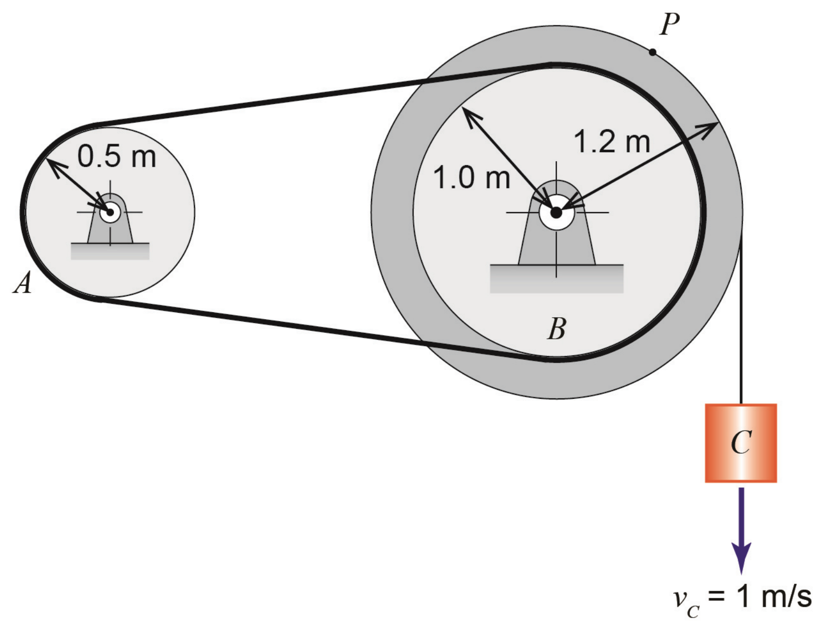
$$\alpha_2 = \frac{r_1}{r_2}\alpha_1$$

Example Problem 4.2-7

In the following figure, mass C is attached to a rope that is wrapped around the outer edge of drum B. As mass C falls, the rope around the drum unrolls without slipping, causing the drum to rotate. This, in turn, causes pulley A to rotate due to the belt connecting the drum's inner radius and the pulley. Consider that mass C is released from rest and reaches a velocity of 1 m/s after it has fallen 30 cm. If the mass accelerates at a constant rate, find for the instant shown

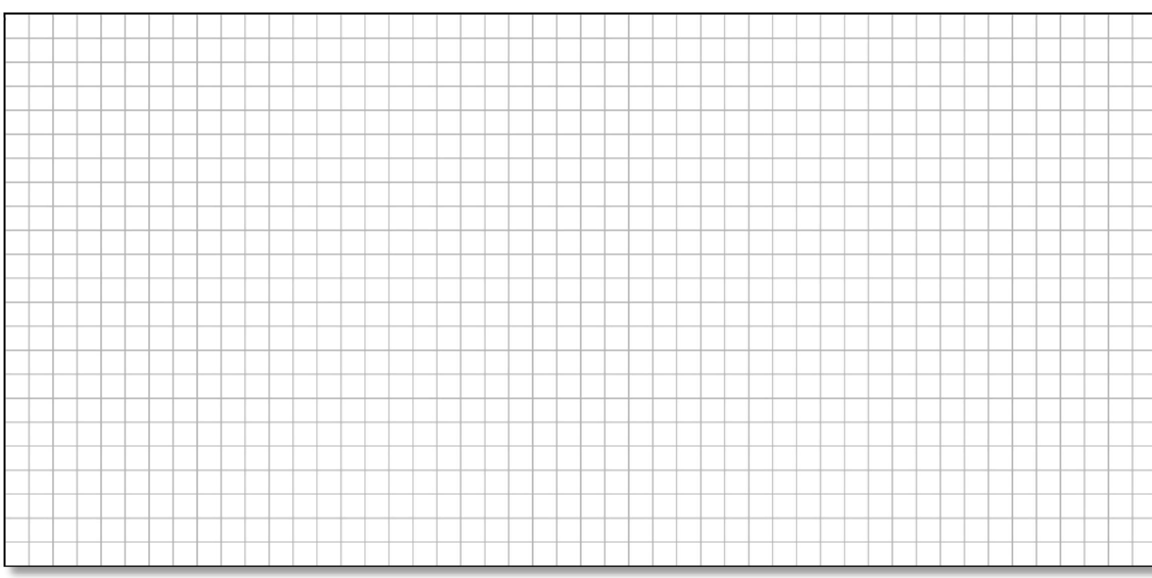
a) the magnitude of the acceleration of point P on the outer surface of the drum,
b) the angular velocity and angular acceleration of drum B, and
c) the angular velocity and angular acceleration of pulley A.

Given:



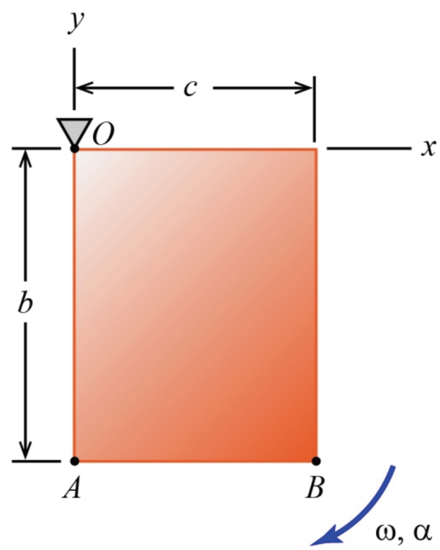
Find:

Solution:



Example Problem 4.2-8

The square plate rotates about the fixed pivot O. At the instant represented, the direction of the angular velocity, ω, and angular acceleration, α, of the plate are shown in the figure. Determine the velocity and acceleration of points A and B in the x-y coordinate frame.

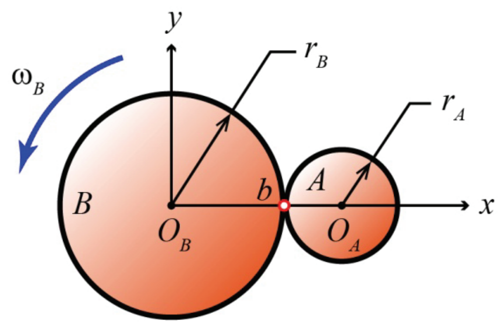


Given:

Find:

Solution:

Solved Problem 4.2-7



Disk B rotates with an angular velocity of $\omega_B = 3t^2$ about a fixed point O_B in the direction shown, where t is in seconds and ω is in revolutions per minute. Disk B rotates disk A through a friction interface. Disk A also rotates about a fixed point O_A. Determine the angular displacement, angular velocity and angular acceleration of disk B after 10 seconds. At the start of motion, point b (on the circumference of disk B) is located at $\theta_B = 0^\circ$. After 10 second, calculate the velocity and acceleration of point b in both n-t and x-y coordinates. Also, after 10 seconds, determine the angular velocity and angular acceleration of disk A assuming there is no slip between the disks. The radii of disk A and B are 4 inches and 8 inches respectively.

<u>Given:</u>

$\omega_B = 3t^2$ rpm
r_B = 8 in
r_A = 4 in
$\theta_{b0} = 0^\circ$
t_f = 10 s

<u>Find:</u>

$\Delta\theta_B$, ω_{Bf}, α_{Bf}
v_{bf}, a_{bf} in n-t and x-y coordinates
ω_{Af}, α_{Af}

<u>Solution:</u>

We are given the angular velocity ω_B of disk B and need to find both the angular displacement $\Delta\theta_B$ and angular acceleration α_B after 10 seconds. We will solve for the unknowns by using the basic relationships shown below. Since ω_B is a function of time, α_B is not constant and we cannot use the constant acceleration relationships.

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt}$$

We will start by finding the final angular velocity. We first need to convert its units to radians per second.

$$\omega_B = 3t^2\,\frac{\text{rev}}{\text{min}}\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right)\left(\frac{\text{min}}{60\,\text{s}}\right) = 0.1\pi t^2\,\frac{\text{rad}}{\text{s}} \qquad \omega_{Bf} = 0.1\pi t_f^{\,2}\,\frac{\text{rad}}{\text{s}}$$

$$\boxed{\omega_{Bf} = 31.4\,\frac{\text{rad}}{\text{s}}}$$

Now that we have ω_B in radians per second, we can calculate the final angular acceleration and angular displacement. The angular acceleration is obtained by differentiating the angular velocity and the angular displacement is obtained by integrating the angular velocity.

$$\alpha_B = \frac{d\omega_B}{dt} = 0.2\pi t \frac{\text{rad}}{\text{s}^2} \qquad \alpha_{Bf} = 0.2\pi t_f \frac{\text{rad}}{\text{s}^2}$$

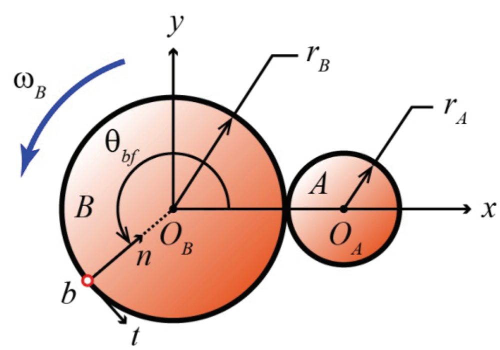
$$\boxed{\alpha_{Bf} = 6.28 \frac{\text{rad}}{\text{s}^2}}$$

$$\omega_B = \frac{d\theta_B}{dt} \quad \rightarrow \quad \int_0^{t_f} \omega_B \, dt = \int_0^{\theta_{Bf}} d\theta_B$$

$$\int_0^{t_f} 0.1\pi t^2 \, dt = \int_0^{\theta_{Bf}} d\theta_B \qquad \left.\frac{0.1\pi t^3}{3}\right|_0^{t_f} = \theta_B\Big|_0^{\theta_{Bf}} \qquad \frac{0.1\pi t_f^3}{3} = \Delta\theta_B$$

$$\Delta\theta_B = 104.72 \,\text{rad}\left(\frac{180^o}{\pi \,\text{rad}}\right) \qquad \boxed{\Delta\theta_B = 6000^o}$$

Now that we know the angular displacement, we can determine the final position of point b by subtracting out the sixteen full rotations that disk B made.



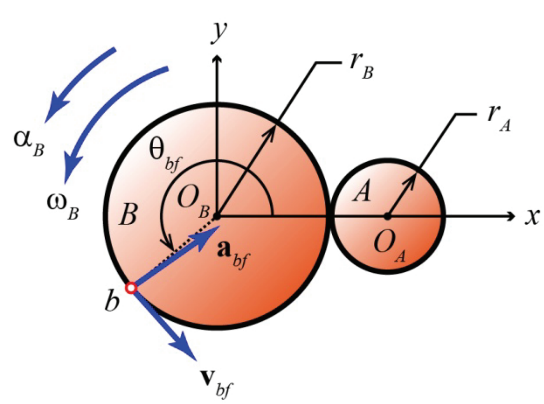
$$\theta_{bf} = 240^o$$

Now that we know the final position of point b, we can calculate its velocity and acceleration. Let's start with the n-t coordinate system.

$$\mathbf{v}_{bf} = r_B \omega_{Bf} \, \mathbf{e}_t \qquad \boxed{\mathbf{v}_{bf} = 251.3 \mathbf{e}_t \frac{\text{in}}{\text{s}}}$$

$$\mathbf{a}_{bf} = r_B \alpha_{Bf} \, \mathbf{e}_t + r_B \omega_{Bf}^2 \, \mathbf{e}_n$$

$$\boxed{\mathbf{a}_{bf} = 50.3 \mathbf{e}_t + 7895.7 \mathbf{e}_n \frac{\text{in}}{\text{s}^2}}$$



Now let's calculate the velocity and acceleration of point b in the x-y coordinate system.

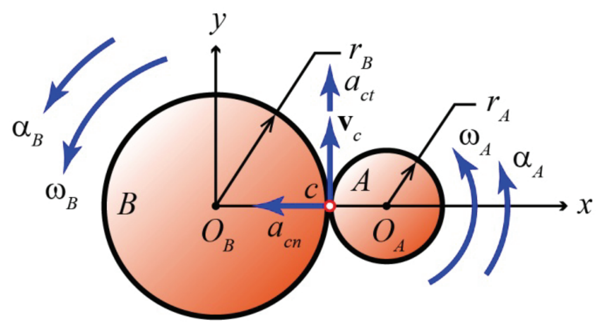
$$\mathbf{v}_b = \boldsymbol{\omega}_{Bf} \times \mathbf{r}_{b/O_B} = \omega_{Bf}\mathbf{k} \times r_B(\cos\theta_{bf}\mathbf{i} + \sin\theta_{bf}\mathbf{j})$$
$$= \omega_{Bf} r_B(\cos\theta_{bf}\mathbf{j} - \sin\theta_{bf}\mathbf{i})$$

$$\boxed{\mathbf{v}_b = 217.7\mathbf{i} - 125.7\mathbf{j}\frac{\text{in}}{\text{s}}}$$

$$\mathbf{a}_{bf} = \boldsymbol{\alpha}_{Bf} \times \mathbf{r}_{b/O} - \omega_{Bf}^2\mathbf{r}_{b/O} = \alpha_{Bf}\mathbf{k} \times \mathbf{r}_B(\cos\theta_{bf}\mathbf{i} + \sin\theta_{bf}\mathbf{j}) - \omega_{Bf}^2 r_B(\cos\theta_{bf}\mathbf{i} + \sin\theta_{bf}\mathbf{j})$$
$$= \alpha_{Bf} r_B(\cos\theta_{bf}\mathbf{j} - \sin\theta_{bf}\mathbf{i}) - \omega_{Bf}^2 r_B(\cos\theta_{bf}\mathbf{i} + \sin\theta_{bf}\mathbf{j})$$

$$\boxed{\mathbf{a}_{bf} = 3987.3\mathbf{i} + 6805.8\mathbf{j}\frac{\text{in}}{\text{s}^2}}$$

Disk B transfers motion to disk A through a friction contact. For every revolution of disk B, disk A needs to complete two revolutions (based on their radii). Therefore, disk A will have an angular velocity and angular acceleration that is twice that of disk B. But, let's confirm this with some calculations.



The point of contact between the two disks must have the same velocity.

$$v_{c,B} = v_{c,A} \qquad r_B\omega_B = r_A\omega_A \qquad \omega_A = \frac{r_B}{r_A}\omega_B$$

$$\boxed{\omega_{Af} = 62.8\frac{\text{rad}}{\text{s}}}$$

The contact point between the two disks must have the same tangential acceleration.

$$a_{c,Bt} = a_{c,At} \qquad r_B\alpha_B = r_A\alpha_A \qquad \alpha_A = \frac{r_B}{r_A}\alpha_B$$

$$\boxed{\alpha_{Af} = 12.6\frac{\text{rad}}{\text{s}^2}}$$

4.3) GENERAL PLANAR MOTION

4.3.1) RELATIVE VELOCITY AND ACCELERATION

Unrestricted, a rigid body can both rotate and translate. In the last section, we looked at the simplified case of pure rotation. We essentially said that the body was restricted to rotate about a fixed axis and, therefore, could only rotate and not translate. In this section, we will look at general planar motion. **General planar motion** allows for simultaneous rotational and translational motion, however, the body is restricted to move within a 2-D plane. This may sound complex, but it can be shown that general planar motion can be considered as a simple superposition of translation plus rotation about a point on the body as shown in Figure 4.3-1.

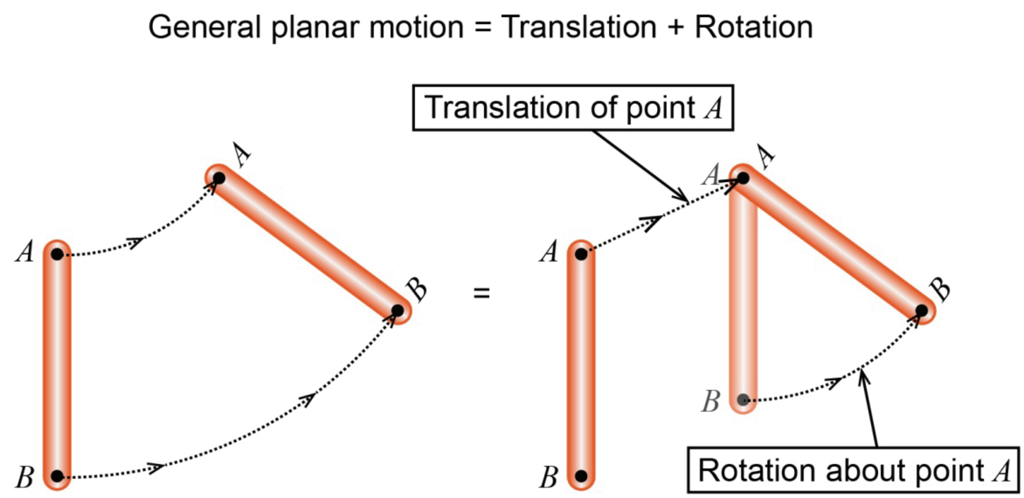
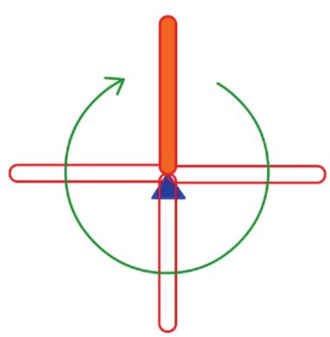


Figure 4.3-1: General planar motion

Conceptual Example 4.3-1

Watch the animation on *Rigid-body Motion* and answer the following questions.



What kind of motion is this?

Conceptual Example 4.3-1 cont.

What kind of motion is this?

What kind of motion is this?

What is general planar motion?

As we have seen, the motion of a rigid body can be considered as a combination of translation and rotation. Consider the bar shown in Figure 4.3-2. In general planar motion, the movement of some point A on the rigid body can be separated into a translational and a rotational part. The motion of point A is equal to the translation of point B plus the rotation of point A about B. The equations for the relative motion of two points on a rigid body are given by Equations 4.3-1 through 4.3-3 when the vectors are given in a non-rotating reference frame. It is always important to remember that when we speak about the motion of a rigid body, the velocity and acceleration of each point on the rigid body may differ if the body is rotating.

Relative velocity: $$\boxed{\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B}} \quad (4.3\text{-}1)$$

$\mathbf{v}_A$, $\mathbf{v}_B$ = absolute velocity of points A and B respectively
$\mathbf{v}_{A/B}$ = velocity of point A relative to point B
ω = angular velocity of the rigid body (rad/s)
$\mathbf{r}_{A/B}$ = position of point A relative to point B

Relative acceleration 2D: $$\boxed{\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} - \omega^2 \mathbf{r}_{A/B}} \quad (4.3\text{-}2)$$

Relative acceleration 3D: $$\boxed{\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B})} \quad (4.3\text{-}3)$$

$\mathbf{a}_A$, $\mathbf{a}_B$ = absolute acceleration of points A and B respectively
$\mathbf{a}_{A/B}$ = acceleration of point A relative to point B
ω = angular velocity of the rigid body (rad/s)
α = angular acceleration of the rigid body (rad/s^2)
$\mathbf{r}_{A/B}$ = position of point A relative to point B

Let's also look at this in another way. To describe the motion of a point on a rigid body that is undergoing general planar motion, we will use the concept of a body-fixed coordinate system. A body-fixed coordinate system is a coordinate system whose origin is attached to and moves with some point on the rigid body. Consider the body shown in Figure 4.3-2. A body-fixed coordinate system (x'-y') is attached to and moves with point B. Point B moves relative to the fixed/inertial coordinate system (x-y) with velocity $\mathbf{v}_B$ and acceleration $\mathbf{a}_B$. Then, consider point B as being on a fixed axis that point A rotates about. In this way, one can envision general planar motion as a superposition of translation ($\mathbf{v}_B$) and rotation ($\mathbf{v}_{A/B}$). Since the distance between points A and B are always constant in a rigid body, the motion of point A relative to point B is circular (i.e. $\boldsymbol{\omega} \times \mathbf{r}_{A/B}$) and the relative velocity is always perpendicular to the line connecting A and B. Note, the general planar motion of the body could be considered as the translation and rotation about any point on the body, not just point B.

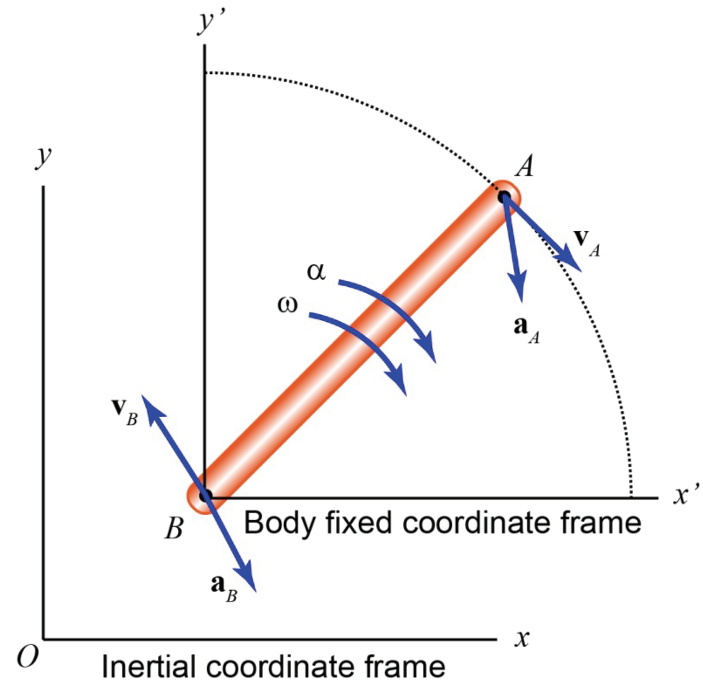


Figure 4.3-2: General planar motion

Example 4.3-2

Referring to Figure 4.3-2, derive the equation for the velocity and acceleration of point A using the following equation for fixed-axis rotation and relative motion.

Fixed-axis rotation equations:

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_{A/O}$$
$$\mathbf{a}_A = \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}$$

Relative motion equations:

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$
$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

What is the velocity and acceleration of point A if point B is fixed and A is rotating about B?

Knowing the velocity and acceleration of point B, what is the absolute velocity and acceleration of A?

Conceptual Example 4.3-3

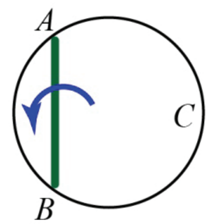
Rod AB slides inside of a stationary ring C. If the rod never loses contact with the ring, what direction is the velocity of end B relative to end A at the instant shown?

a) → b) ←

c) ↓ d) ↑

e) ↗ f) ↖

g) ↙ h) ↘



4.3.2) MECHANISMS

Many mechanical systems involve rigid bodies connected together in various ways to convert motion into different forms. For example, the linear motion of a piston in an automobile engine is transferred into rotary motion as shown in Figure 4.3-3. Another example is the bicycle mechanism that converts your rotary motion into straight line motion as shown in Figure 4.3-4. Other mechanisms, such as gears, only change the speed of rotation (see Figure 4.3-5).

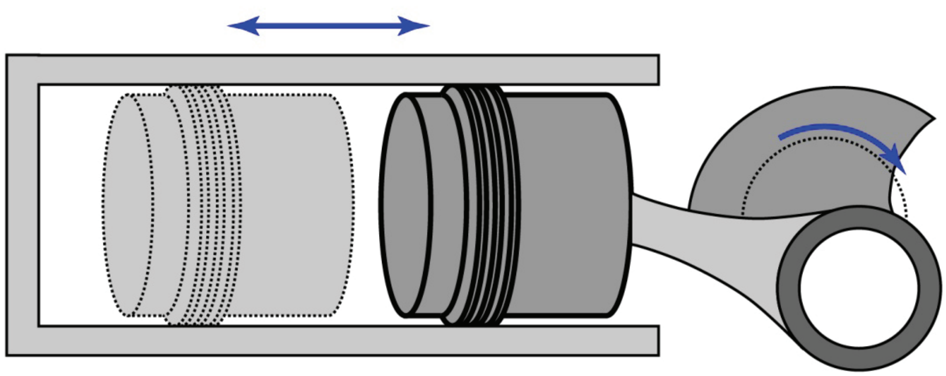


Figure 4.3-3: Piston mechanism

Figure 4.3-4: Bicycle sprocket and chain mechanism

Figure 4.3-5: Gear mechanism

4.3.3) CONSTRAINTS

We examined constrained motion in the chapter on *kinematics of particles - curvilinear motion* and saw how various constraints affected the motion of particles. In this section, we will extend this knowledge and look at some examples of the constrained motion of rigid bodies.

Many mechanisms that you will see in real life have constrained motion. For example, a piston is constrained to move in the direction of the cylinder. A car is constrained to move along the road profile on which it is driving (that is, unless you are Steve McQueen in the film Bullet.) If we can identify and characterize each constraint in a mechanism, it will aid in its analysis.

Many times complex systems can be modeled as simple mechanisms or connecting elements. Examples of some types of mechanisms that constrain or connect bodies are linear bearings, joints, and surface contacts.

4.3.4) LINEAR BEARINGS / COLLARS

Linear bearings and collars are sleeves that fit around or on shafts as described in the chapter on *kinematics of particles - curvilinear motion*. If a rod or linkage is connected to the bearing or collar, the connection point is constrained to move along the direction of the shaft. The rotational constraints imposed on the rod depend on the joint used to connect the rod and collar. A good example of bearing/collar use is a weight machine, such as, the one shown in Figure 4.3-6. The collars help guide the weights and bars so that you can concentrate on lifting the weights without having to worry about balancing the weights.

Figure 4.3-6: Exercise machine

4.3.5) SLOTS

A slot constrains the motion of a pin to move in the direction of the slot. The velocity of the pin is always tangent to the path of the slot as shown in Figure 2.8-3. Examples of constrained slot motion are coin slots, such as in a coin operated laundry, and a Geneva wheel mechanism (shown in Figure 4.3-7). The coin slot constrains the coins to move along a certain path, however; the coins are free to rotate. The Geneva wheel transfers rotational motion between two shafts at prescribed intervals. The interval frequency depends on the wheel's design and the angular velocity of the driving shaft.

Figure 4.3-7: Geneva wheel

4.3.6) JOINTS

Joints can restrict the motion of mechanisms that they are attached to from 1 to 6 degrees of freedom. In total, a rigid body has 6 degrees of freedom or 6 ways in which it can move. A completely unconstrained body can translate in the x-, y- and z-directions and can rotate about the x-, y- and z-directions for a total of 6 degrees of freedom.

There are four main types of joints: a clamped or welded joint, a pinned or hinged joint, a ball and socket joint and a roller joint. A clamped joint restricts all 3 directions of translation and all 3 directions of rotation. This means that the end condition attached to a clamped joint does not move. A pinned joint restricts all 3 directions of translation at the joint and allows for free rotation in one direction about the joint. A ball and socket restricts all 3 directions of translation and allows for free rotation in all three directions about the joint. The last type of joint is a roller joint. This joint may allow up to 3 directions of translational motion and all directions of rotation. All types of joints are illustrated in Figure 4.3-8.

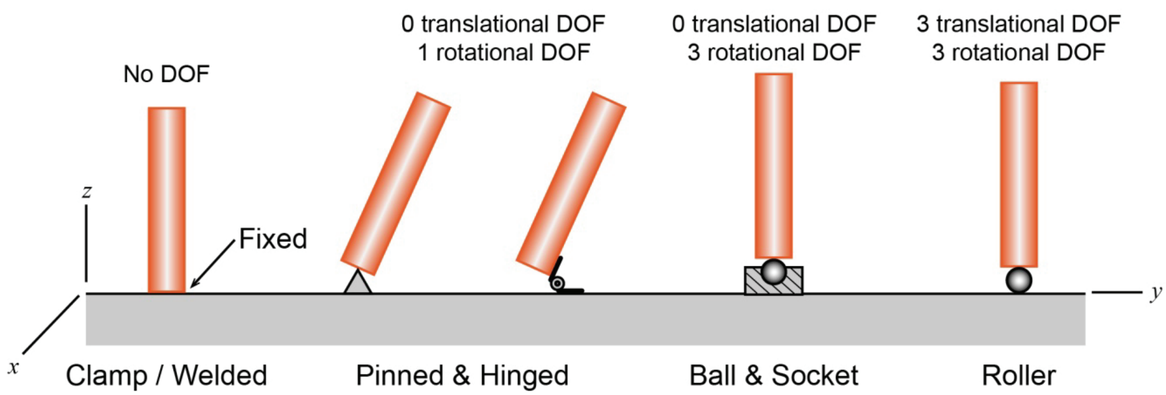


Figure 4.3-8: Types of joints

4.3.7) SURFACE CONTACTS

Surface contacts constrain the motion of some point on the rigid body to the profile of the surface. For example, if a car is riding along a road, the tires are forced to move up and down with the undulations of the road (see Figure 4.3-9).

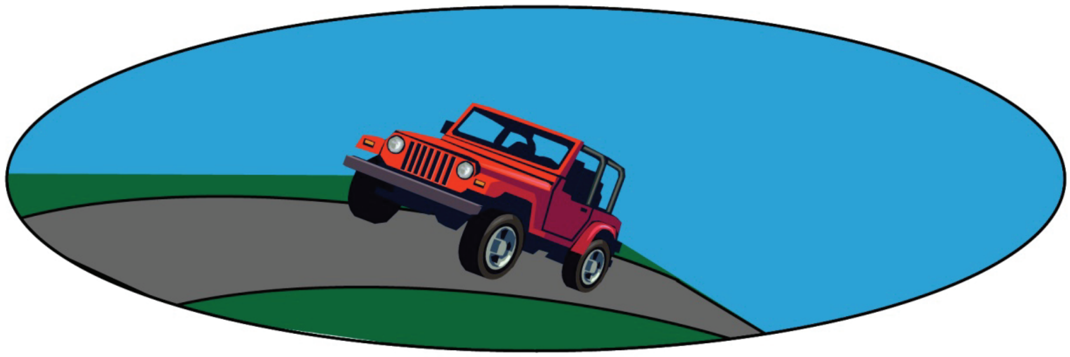


Figure 4.3-9: Car and road

4.3.8) ANALYZING MECHANISMS

We can analyze the planar motion of rigid-body mechanisms using the relative velocity and acceleration equations given in Equations 4.3-1 and 4.3-2. We will first present an overall strategy for analyzing these systems before examining some more commonly encountered examples.

Step 1: Get familiar with the system

- Determine if there are any fixed axes/points of rotation in the system or any points of known velocity.
- Determine if there are any geometric constraints. For example, a point sliding in a channel or along a surface, or a rolling contact.
- Assign angular velocity directions. Your assignment may be incorrect initially, but if you are consistent in your analysis, this error will be discovered in your results.

Step 2: Analyze the velocities

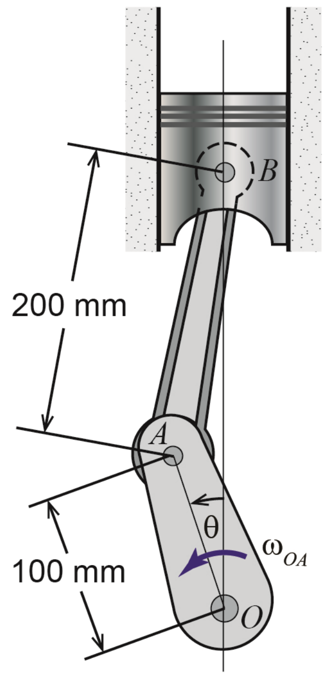
- Start at a position that is fixed or that has known velocity and work outward.
- Sometimes two different relationships can be found for the motion of a single point by starting from different points and meeting in the middle.
- Apply known information and geometric constraints to eliminate unknowns.

Step 3: Analyze the accelerations

- Repeat the process presented for the velocity. Typically the expressions used to calculate the accelerations of the system depend on the velocities of the system.

Example Problem 4.3-4

Consider the following diagram of a piston assembly in an internal combustion engine. The crankshaft OA is rotating at a constant rate of 2000 rpm counter-clockwise. If the piston is at top dead center position ($\theta = 0^{\circ}$), determine (a) the velocity and acceleration of piston head B, (b) the angular velocity of connecting rod AB, and (c) the angular acceleration of connecting rod AB.



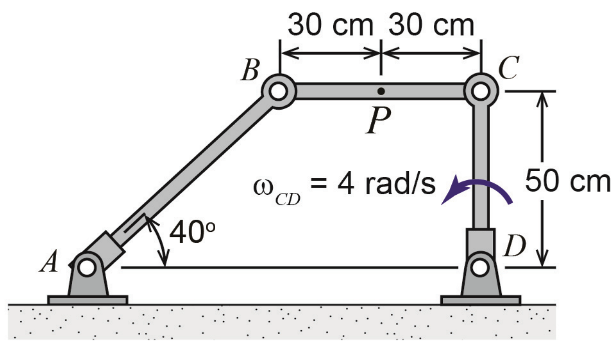
Given:

Find:

Solution:

Example Problem 4.3-5

Consider the four-bar linkage shown. If link CD is being driven with an angular velocity of 4 rad/s, determine the velocity of point P on link BC and the angular velocity of link AB at the instant shown.



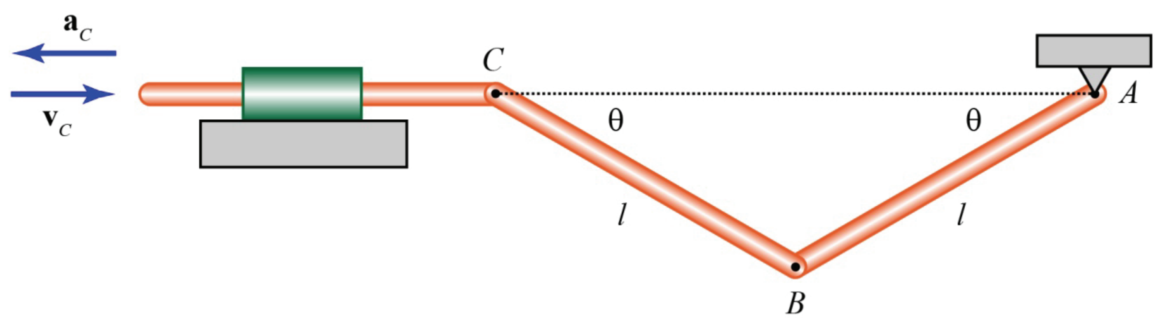
Given:

Find:

Solution:

Solved Problem 4.3-6

The three-bar mechanism shown is put into motion by an actuator. The actuator applies a known velocity ($\mathbf{v}_C$) and acceleration ($\mathbf{a}_C$) to the end of link C. Link C is constrained to move horizontally by a collar. Links AB and BC are the same length and end A is fixed by a pin joint. Determine the angular velocity and angular acceleration of links AB and BC when the links make an angle of θ with the horizontal, as shown.

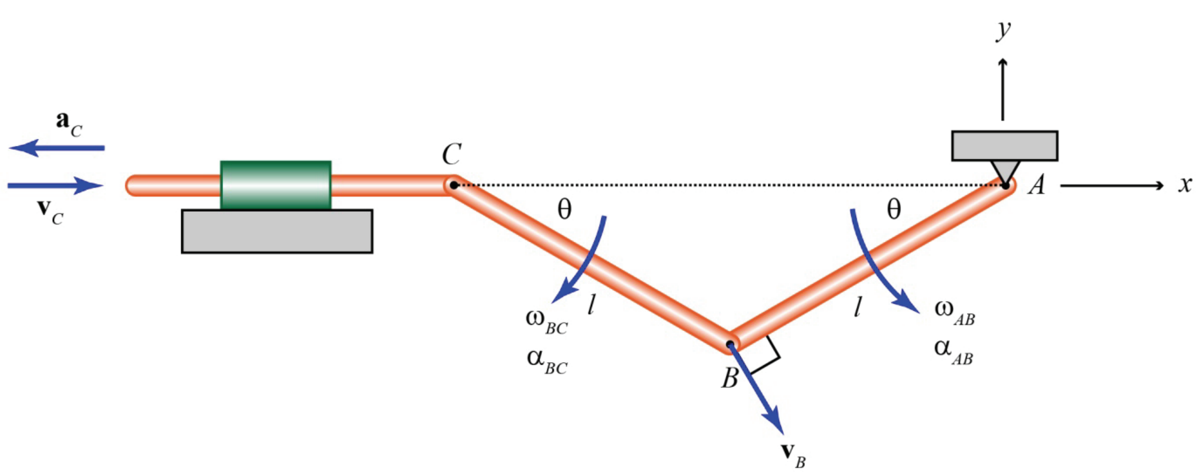


Given: $\mathbf{v}_C$, $\mathbf{a}_C$
The length of link AB and BC
θ

Find: ω_{AB}, ω_{BC}
α_{AB}, α_{BC}

Solution:

Step 1: Get familiar with the system. Assign all linear velocity, angular velocity and angular acceleration directions. If you are uncertain which direction the angular velocity or angular acceleration are in, that's ok. Take a guess. If you are incorrect, the angular velocity or angular acceleration result will be negative letting you know that you guessed wrong. Also, you will need to add a coordinate system is one is not specified for you.



Step 2: Analyze the velocities. Our strategy for solving this problem is to work from a known velocity (joint A) to joint C. We can then use the constraint applied to link C to determine the unknowns.

Let's look at link AB.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = 0 + \omega_{AB}\mathbf{k} \times l(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) = \omega_{AB} l(-\cos\theta\mathbf{j} + \sin\theta\mathbf{i})$$

Now let's look at link BC and get an expression for the velocity of C.

$$\begin{aligned}\mathbf{v}_C &= \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = \mathbf{v}_B - \omega_{BC}\mathbf{k} \times l(-\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = \mathbf{v}_B + \omega_{BC} l(\cos\theta\mathbf{j} + \sin\theta\mathbf{i}) \\ &= l\left[(\omega_{AB} + \omega_{BC})\sin\theta\mathbf{i} + (-\omega_{AB} + \omega_{BC})\cos\theta\mathbf{j}\right]\end{aligned}$$

We will use the fact that $\mathbf{v}_C$ does not have a $\mathbf{j}$ component to solve for the unknowns.

$$\mathbf{j}: 0 = l(-\omega_{AB} + \omega_{BC})\cos\theta \qquad \omega_{AB} = \omega_{BC} = \omega$$

$$\mathbf{i}: v_C = l(\omega_{AB} + \omega_{BC})\sin\theta = 2\omega l \sin\theta \qquad \boxed{\omega = \frac{v_C}{2l\sin\theta}}$$

Step 3: Analyze the accelerations. We will use a similar procedure to solve for the angular accelerations.

$$\begin{aligned}\mathbf{a}_B &= \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A} = 0 + \alpha_{AB}\mathbf{k} \times l(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) - \omega^2(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) \\ &= \alpha_{AB} l(-\cos\theta\mathbf{j} + \sin\theta\mathbf{i}) - \omega^2(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) = l\left[(\alpha_{AB}\sin\theta + \omega^2\cos\theta)\mathbf{i} + (-\alpha_{AB}\cos\theta + \omega^2\sin\theta)\mathbf{j}\right]\end{aligned}$$

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega^2\mathbf{r}_{C/B} = \mathbf{a}_B - \alpha_{BC}\mathbf{k} \times l(-\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) - \omega^2(-\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \\ &= \mathbf{a}_B + \alpha_{BC} l(\cos\theta\mathbf{j} + \sin\theta\mathbf{i}) - \omega^2(-\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \\ &= l\left[(\alpha_{AB}\sin\theta + \omega^2\cos\theta + \alpha_{BC}\sin\theta + \omega^2\cos\theta)\mathbf{i} + (-\alpha_{AB}\cos\theta + \omega^2\sin\theta + \alpha_{BC}\cos\theta - \omega^2\sin\theta)\mathbf{j}\right] \\ &= l\left[(\alpha_{AB}\sin\theta + 2\omega^2\cos\theta + \alpha_{BC}\sin\theta)\mathbf{i} + (-\alpha_{AB}\cos\theta + \alpha_{BC}\cos\theta)\mathbf{j}\right]\end{aligned}$$

$$\mathbf{j}: 0 = l(-\alpha_{AB}\cos\theta + \alpha_{BC}\cos\theta) \qquad \alpha_{AB} = \alpha_{BC} = \alpha$$

$$\mathbf{i}: a_C = l(2\alpha\sin\theta + 2\omega^2\cos\theta) \qquad \boxed{\alpha = \frac{a_C - 2l\omega^2\cos\theta}{2l\sin\theta}}$$

4.4) INSTANTANEOUS CENTER OF ZERO AND KNOWN VELOCITIES

When analyzing the motion of rigid bodies, it is very helpful if we can identify a point of known velocity. This velocity gives us a starting point and helps us solve for relevant unknown information. We have already discussed some instances where points have known velocities. A fixed axis/point of rotation has a known velocity that is equal to zero. If the end of a link is constrained to slide or move along another component of the mechanism, then we may not know the velocity. But, we know something about its direction. In this section, we will focus on how to identify other points of known velocity. The first type of point we will look at is a specialized case called the *instantaneous center of zero velocity* or *IC.* Then, we will look at special cases, such as rolling, where points of known velocities can be determined by looking at the motion of other components in the system.

4.4.1) INSTANTANEOUS CENTER OF ZERO VELOCITY

The **instantaneous center of zero velocity** (IC), for a rigid body, is a point on or off the rigid body that has zero velocity for an instant. The IC may be treated as a fixed axis of rotation for that instant. Even though the IC has zero velocity at a particular instant, it usually has non-zero acceleration. This means that the location of the IC may change with time. Utilizing the IC of a rotating body often saves calculation effort. If the location of the IC is known, solving for the velocity of any other point on the rigid body is simplified as shown in Equation 4.4-1.

Velocity relative to the IC: $$\boxed{\mathbf{v}_P = \mathbf{v}_{IC} + \mathbf{v}_{P/IC} = \boldsymbol{\omega} \times \mathbf{r}_{P/IC}} \quad (4.4\text{-}1)$$

$\mathbf{v}_P$ = velocity of point P
$\mathbf{v}_{IC}$ = velocity of the IC
$\boldsymbol{\omega}$ = angular velocity
$\mathbf{r}_{P/IC}$ = position of point P relative to the IC

Not only can the IC simplify the necessary calculations, but it can also help in visualizing the magnitude and direction of the velocity for any point on the rigid body. The direction of the velocity of any point on the body is perpendicular to the line connecting the IC and the point in question. The magnitude of the velocity is equal to the body's angular velocity multiplied by this length. For example, Figure 4.4-1 shows a disk rolling without slip on a stationary ground. The IC is located at the point where the disk contacts the ground. This is true because there cannot be a discontinuity between the velocity of the ground and the velocity of the rolling body if there is no slip. Which means that, if the ground has zero velocity, the contacting point will also have zero velocity. This can be extended to say that whatever velocity the ground has (zero or non-zero) the contacting point will also have if there is no slip. Referring to Figure 4.4-1, the velocity of point A is perpendicular to the line connecting the IC and point A and its speed is equal to $\omega r_{A/IC}$. Also, the speed of point B is greater than the speed of point A because it is farther away from the IC.

When trying to identify an IC for a given situation, keep the following two things in mind.

1. The velocity at any point on an object is always perpendicular to its position vector as measured from the IC.
2. The velocities are proportional to the distance that the point is from the IC.

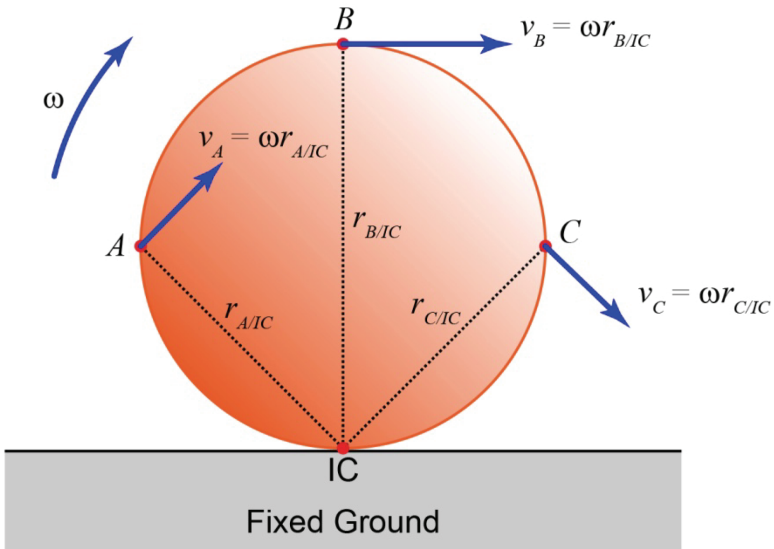
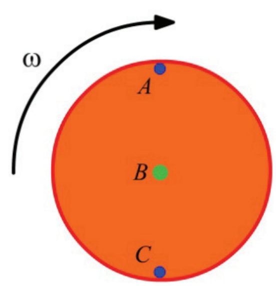


Figure 4.4-1: Visualizing velocities using the IC

Example 4.4-1

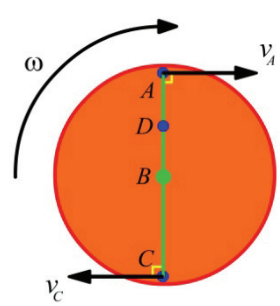
Watch the *IC* animation and answer the following questions.

1) What direction is the velocity of point A?



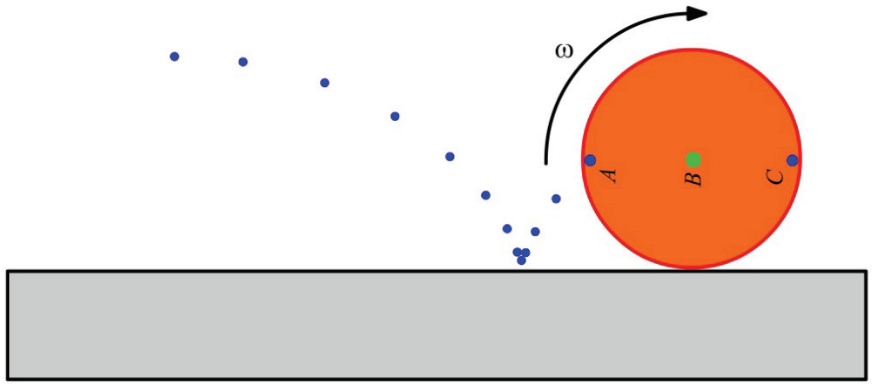
2) What is the velocity of B?

3) Is the velocity of point D smaller or larger than the velocity of point A?



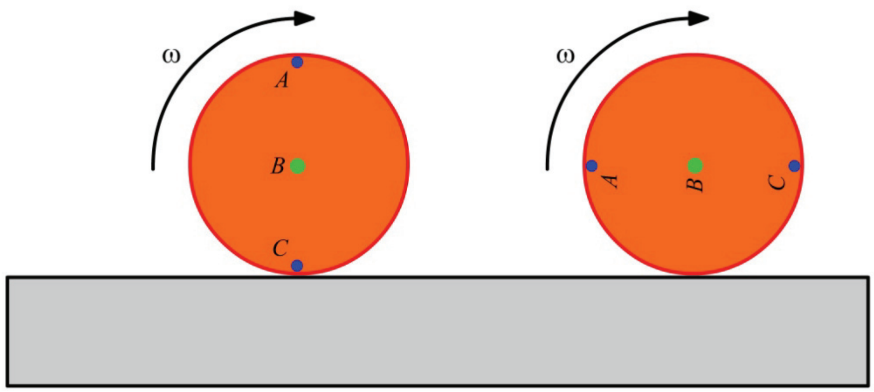
Example 4.4-1 cont.

4) What direction is the velocity of point A?



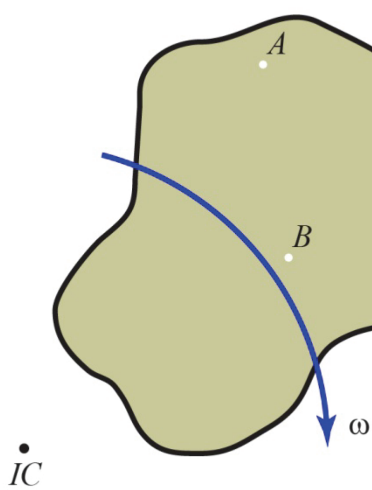
5) Where is the IC at the instant shown?

6) In which case does point A have the larger velocity?



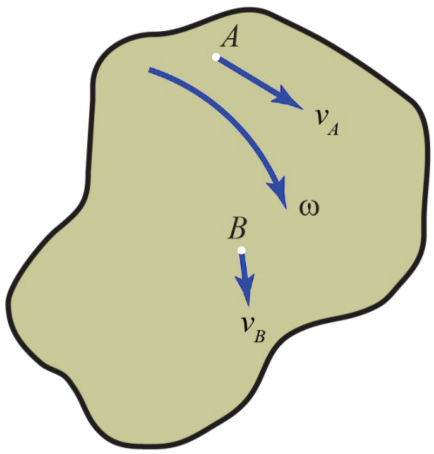
Example 4.4-2

Draw the direction of the velocity of point A and point B. Which velocity is larger?



Example 4.4-3

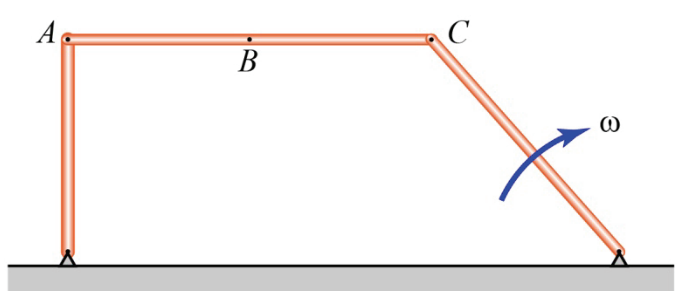
Given the velocities of point A and B, locate the IC.



Example 4.4-4

A four-bar linkage is set into motion as shown. It has pin joints at A, C and where it connects to the base. Following the steps below, sketch the directions of the velocities for points A, B and C.

1. Locate points of known velocity.
2. Identify the directions of the angular velocities?
3. What are the directions of $\mathbf{v}_A$ and $\mathbf{v}_C$?
4. Identify the IC for link AC.
5. What is the direction of $\mathbf{v}_B$?



Example 4.4-5

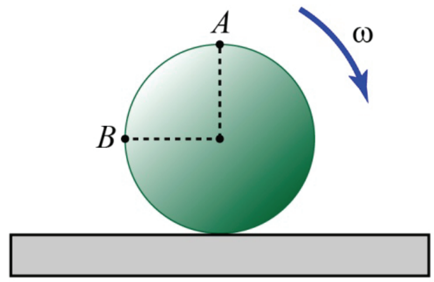
A disk rolls, without slip, on a stationary ground. What is the velocity direction of point B relative to point A?

a) → b) ←

c) ↓ d) ↑

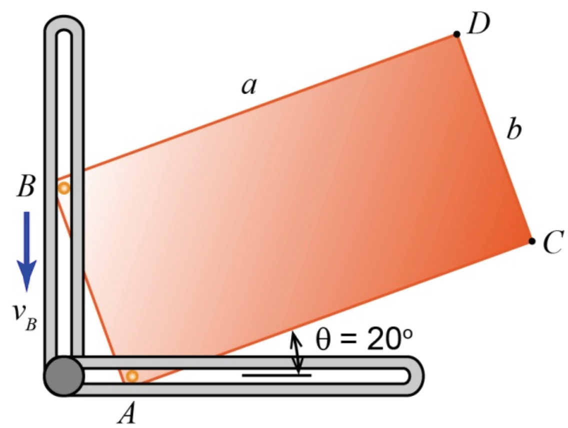
e) ↗ f) ↖

g) ↙ h) ↘



Example Problem 4.4-6

The rectangular plate is confined by rollers within slots at A and B. When $\theta = 20^{\circ}$, point B is moving at $v_B = 10$ m/s. Determine the velocity of points C and D at this instant. The dimensions of the plate are $a = 100$ mm and $b = 50$ mm.



Given:

Find:

Solution:

4.4.2) ROLLING

A specific type of motion that exemplifies how rigid bodies can rotate and translate simultaneously is rolling. This type of motion is relevant to many types of real life problems, such as, tires on cars, motorcycles and bicycles. This relates to any mechanism that transfers motion through a rolling contact to create linear motion. There are two cases of motion that may occur for rolling contacts. Each case requires a completely different set of equations for analysis. First, a wheel or other rolling object can roll without slip. In this situation, we can identify the IC as being the point of the wheel that touches the ground. The friction that occurs between the ground and the rolling contact, in this case, is called static friction. Second, the wheel rolls with slip. Imagine that you are driving your car on an icy road and you slam on your brakes and lose traction. This is what happens in the second case. There is relative motion between the wheel and the ground. This means we cannot identify the IC as the contact point between the wheel and the ground. The friction between the two bodies, in this case, is called kinetic friction. It is important to understand and be able to analyze both cases, but in this section we will only be looking at the no slip case. Rolling with slip will be addressed in detail in the *Rigid Body Newtonian Mechanics* chapter.

What is the difference between a wheel that rolls without slip and one that rolls with slip?

In the case where we have rolling without slip, you can imagine that the point of the body that has contact with the ground does not move relative to the ground. The contact point does not slide, rather it just touches the ground before it rotates away. The next point on the body subsequently takes its position in contact with the ground. For this situation, we can generate equations that express the translational motion of the geometric center of the body (O) as shown in Figure 4.4-2.

Imagine, for example, that a rope is wrapped around the circumference of the body. As the body rolls, this rope is unwound and laid upon the ground. The length of rope laid down represents the displacement of the body's center. The displacement is equal to the arc length corresponding to the angle the body has rolled through (Equation 4.4-2). Taking the time derivative of this expression for displacement provides equations for the translational velocity and acceleration of the body's geometric center O (Equations 4.4-3 and 4.4-4).

Distanced traveled by body center O: $\boxed{s_O = r\theta}$ (4.4-2)

s_O = distance traveled by the center of the rolling body
r = radius
θ = angle the body has rolled through (rad)

Speed of body center O: $\boxed{v_O = r\omega}$ (4.4-3)

Acceleration magnitude of body center O: $\boxed{a_O = r\alpha}$ (4.4-4)

v_O = magnitude of the velocity of the rolling body's center
a_O = magnitude of the acceleration of the rolling body's center
r = radius
ω = angular velocity (rad/s)
α = angular acceleration (rad/s^2)

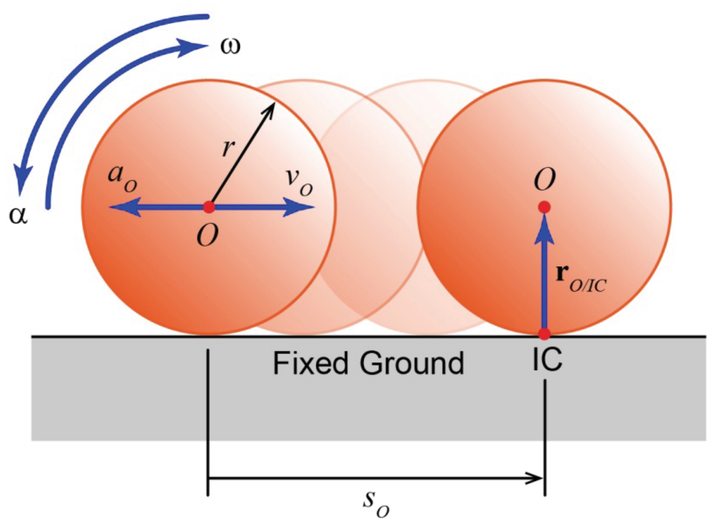


Figure 4.4-2: Rolling without slip

We can also derive the velocity and acceleration for the center of a body rolling without slip in a slightly different way. Consider the relative velocity equation that we use to analyze rigid-body motion ($\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$). If we apply this equation between the body center O and the IC, we get the following equation (see Figure 4.4-2).

$$\mathbf{v}_O = \mathbf{v}_{IC} + \boldsymbol{\omega} \times \mathbf{r}_{O/IC}$$

The velocity of the IC is zero, therefore, we end up with the velocity of the body center given by Equation 4.4-5.

Velocity of body center O: $\boxed{\mathbf{v}_O = \boldsymbol{\omega} \times \mathbf{r}_{O/IC}}$ (4.4-5)

$\mathbf{v}_O$ = velocity of the body center O
$\mathbf{r}_{O/IC}$ = position of the body center O relative to the IC
$\boldsymbol{\omega}$ = angular velocity (rad/s)

If we take the derivative of Equation 4.4-5, we obtain an equation for the acceleration of the body center.

$$\mathbf{a}_O = \frac{d\mathbf{v}_O}{dt} = \frac{d(\boldsymbol{\omega} \times \mathbf{r}_{O/IC})}{dt} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_{O/IC} + \boldsymbol{\omega} \times \frac{d\mathbf{r}_{O/IC}}{dt}$$

Looking at the above derivative, it is easy to see that the derivative, with respect to time of the angular velocity, is the angular acceleration (i.e. $d\omega/dt = \alpha$). But, what is the derivative of the position vector of O with respect to the IC ($d\mathbf{r}_{O/IC}/dt = ?$)? If you think about it, the size of $\mathbf{r}_{O/IC}$ never changes because the distance between O and IC never changes. It is also true that, the direction of $\mathbf{r}_{O/IC}$ never changes because the IC is always the contact point and the body center O is always directly above the contact point. Therefore, the derivative of $\mathbf{r}_{O/IC}$ with respect to time is zero ($d\mathbf{r}_{O/IC}/dt = 0$). The acceleration of the body center is then given by Equation 4.4-6.

Acceleration of body center O: $$\boxed{\mathbf{a}_O = \boldsymbol{\alpha} \times \mathbf{r}_{O/IC}} \quad (4.4\text{-}6)$$

$\mathbf{a}_O$ = acceleration of the body center O
$\mathbf{r}_{O/IC}$ = position of the body center O relative to the IC
α = angular acceleration (rad/s^2)

Equations 4.4-2 through 4.4-6 are only valid for a body that is rolling without slip. You can see from the above derivation that if the body was slipping, the contact point would not be an IC. The assumption of v_{IC} = 0 is essential to the first step in the derivation of the equations. Without this assumption, the above equations do not hold.

Once the motion of the center of a rolling body is known, then the translational motion of other points can be determined. This is possible through the use of the relative velocity and acceleration equations. For example, if it were desired to determine the velocity and acceleration of point A on a rolling body as shown in Figure 4.4-3, we could use the following equations.

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A/O} = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{A/O}$$
$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O} = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}$$

Conversely, if information about the body's rotation is unknown, we can use the velocity and acceleration of the body center to determine the angular velocity and acceleration as shown below.

$$\omega = \frac{v_O}{r} \qquad\qquad \alpha = \frac{a_O}{r}$$

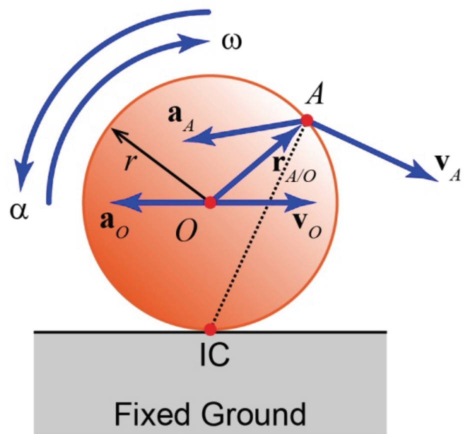
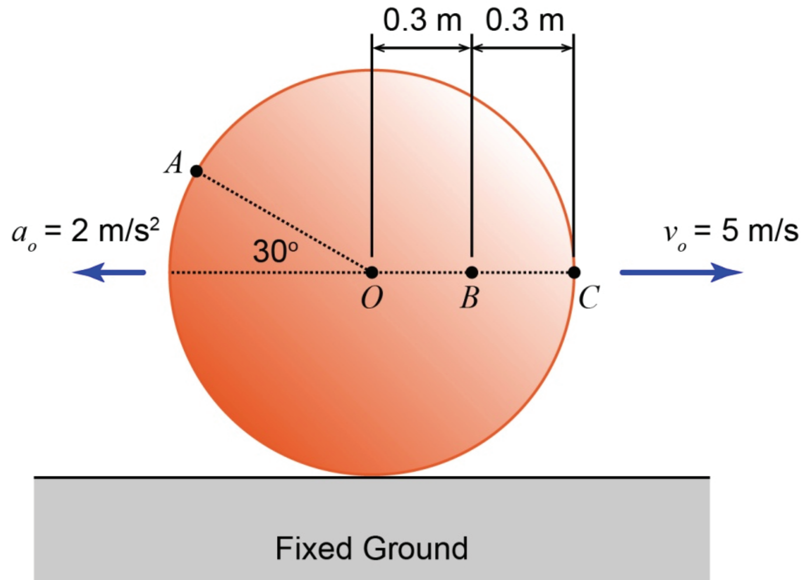


Figure 4.4-3: Translational motion of points on a rolling body

Example Problem 4.4-7

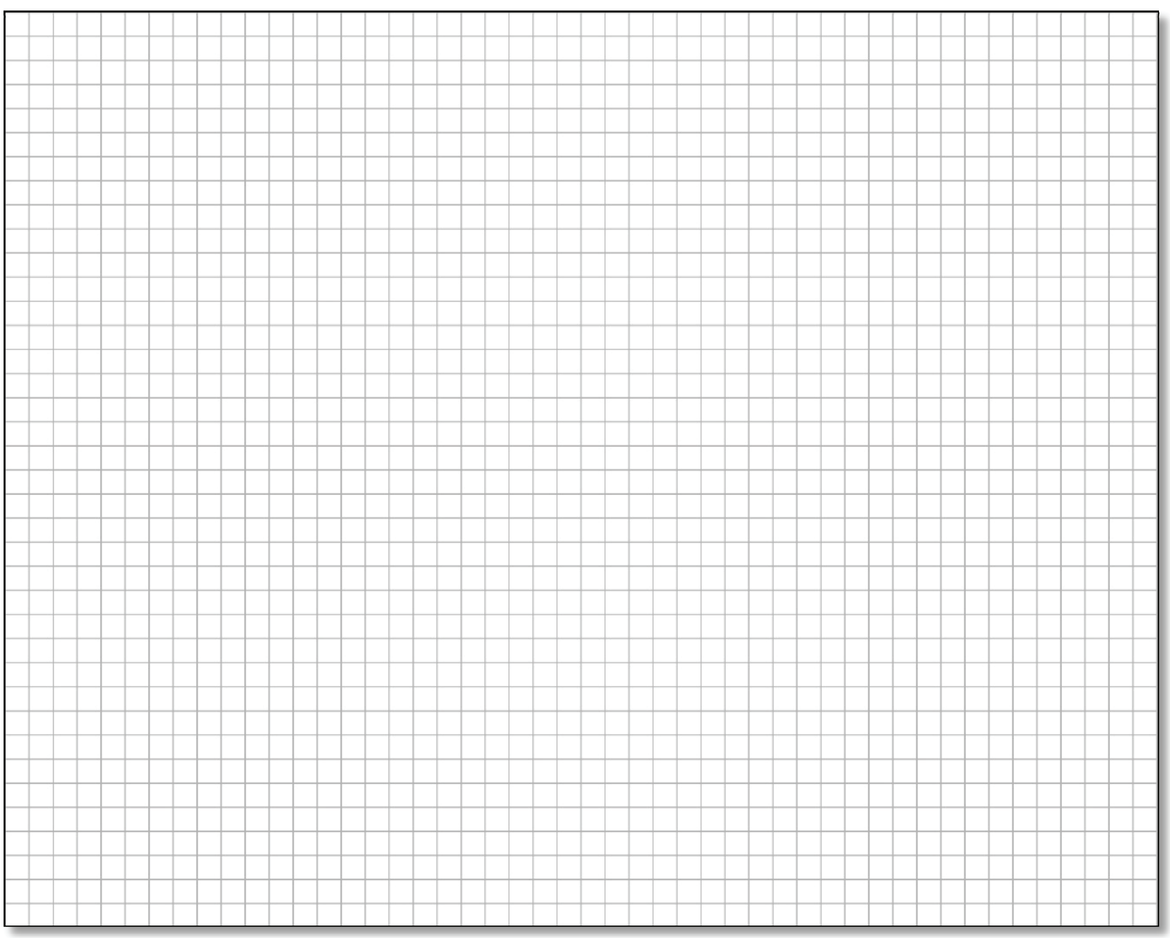
Center O of the disk has the velocity and acceleration shown in the figure. If the disk rolls without slipping on the horizontal surface, determine the velocity and acceleration of points A, B and C for the instant represented.



Given:

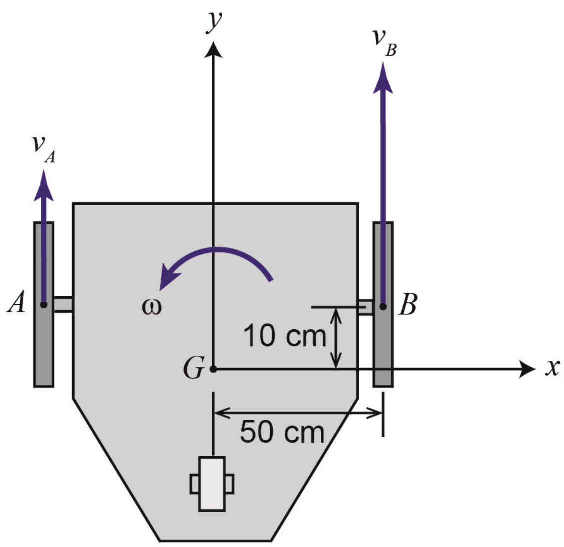
Find:

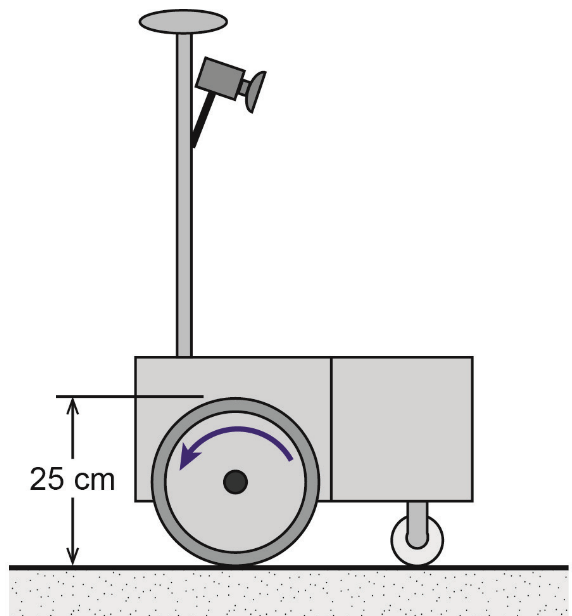
Solution:



Example Problem 4.4-8

Shown below, the autonomous ground robot is geometrically symmetric about the y axis. The two large wheels of the robot are driven by their own electric motors that cause the wheels to turn, propelling the vehicle forward. The vehicle is able to turn by commanding each of the two wheels to rotate at different speeds. If the left wheel is commanded to rotate at 4 rad/sec and the right wheel is commanded to rotate at 6 rad/sec, determine for the instant shown (a) the translational velocity of the center of the left wheel (point A) and the translational velocity of the center of the right wheel (point B), (b) the translational velocity of the vehicle's mass center G, and (c) the vehicle's overall angular velocity $\boldsymbol{\omega}$. Assume that each of the two wheels roll without slip.

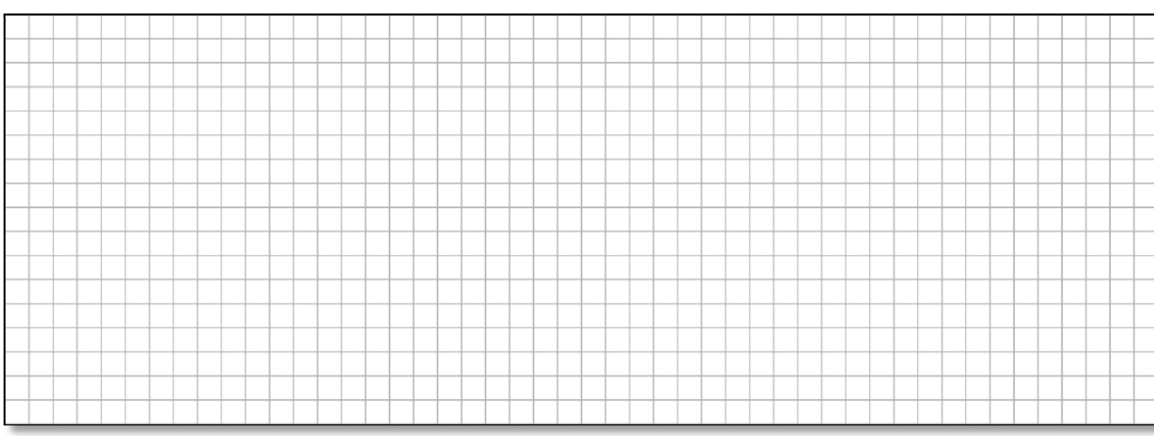




Given:

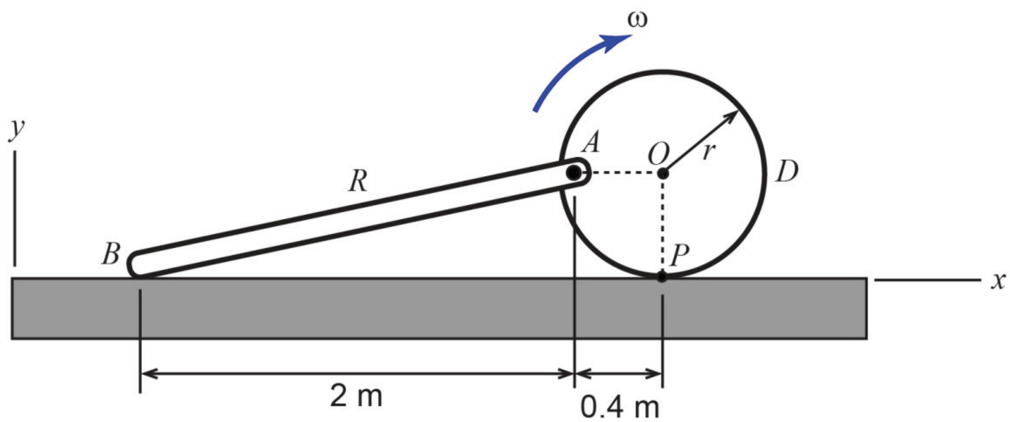
Find:

Solution:



Example Problem 4.4-9

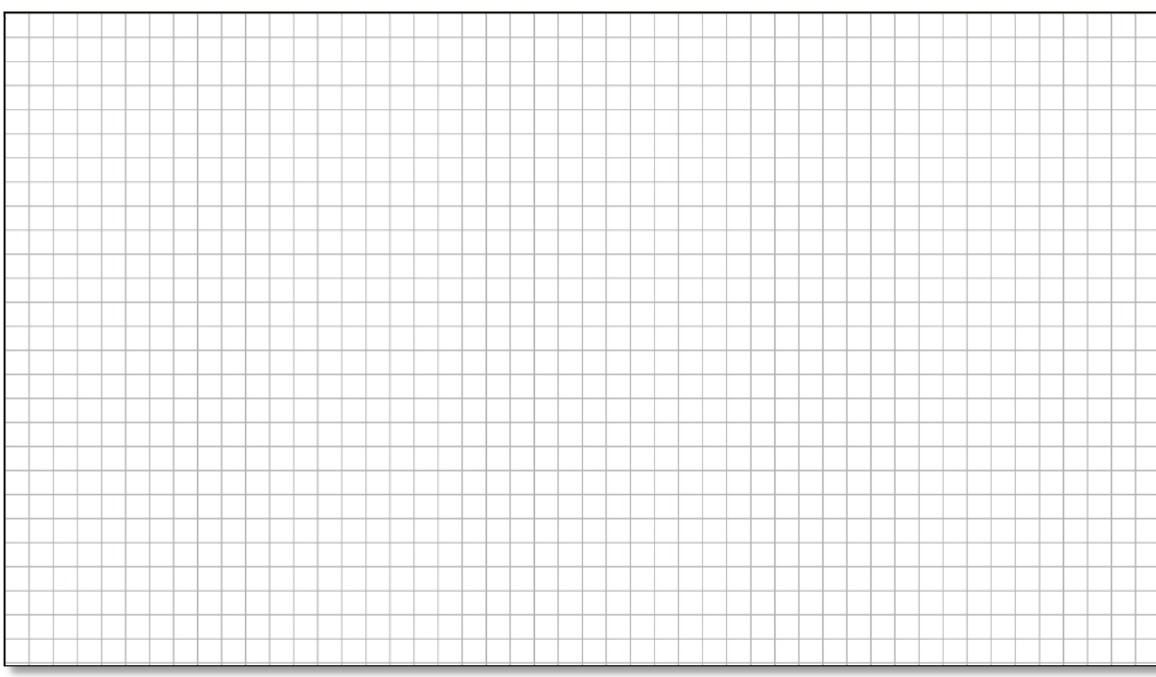
Disk D, of radius r = 0.5 m, rolls without slipping on a horizontal floor with a constant clockwise angular velocity of ω = 3 rad/s. Rod R, is hinged to D at A, and the end of the rod B, slides along the floor. Determine the angular velocity of R when the line OA joining the center of the disk to the hinge at A is horizontal as shown.



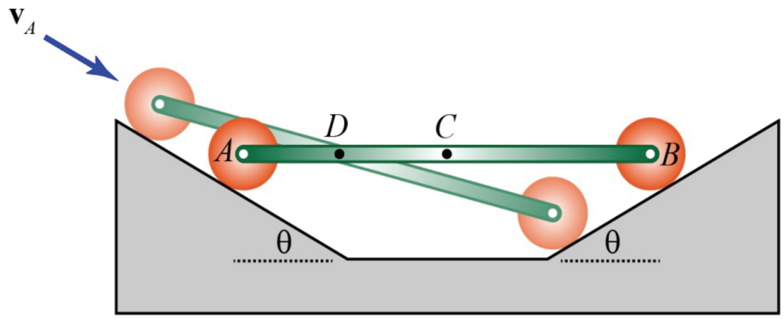
Given:

Find:

Solution:



Solved Problem 4.4-10

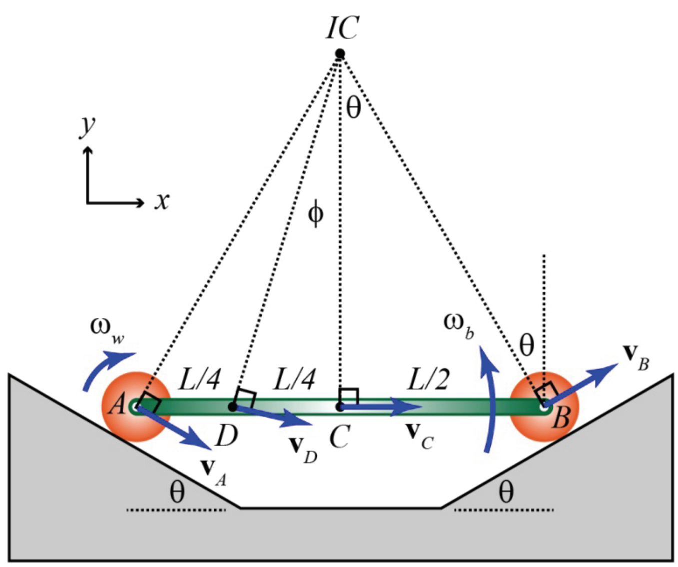


Wheels of radius 0.17 m are attached to the end of a bar of length 1 m. Each wheel runs on a 30-degree incline without slip. The velocity of the wheel attached to end A is a constant 0.5 m/s, in the direction shown. When the bar is horizontal, determine the angular velocity of the bar and wheels. Also, determine the velocity of points C and D. Point C is located at the center of the bar and point D is located half way between point C and point A.

Given: L = 1 m
r = 0.17 m
$\theta = 30^o$
v_A = 0.5 m/s
No slip

Find: ω_{bar}, ω_{wheel}
$\mathbf{v}_C$, $\mathbf{v}_D$

Solution:



We will solve this problem through the use of the bar's instantaneous center of zero velocity (IC). Let's draw the location of the bar's IC and use this to determine the directions of the velocities of various points of interest. If we know the direction of two velocities, we can locate the IC. We know the directions of v_A and v_B since they are parallel to their respective inclines. Since the IC gives us the directions of the unknown velocities, we can use the scalar version of the velocity equation to calculate their sizes (i.e. $v_P = r_{P/IC}\,\omega$). Before we can calculate the unknown velocities, we need to determine the bar's angular velocity.

$$v_A = r_{A/IC}\omega_{bar} = \frac{L}{2\sin\theta}\omega_{bar}$$

Note: The triangle A-B-IC is an equilateral triangle. This would make $r_{A/IC}$ equal to L. However, the calculations will be carried out regardless of this fact.

$$\omega_{bar} = \frac{2v_A \sin\theta}{L} \qquad \boxed{\boldsymbol{\omega}_{bar} = 0.5\mathbf{k}\,\frac{\text{rad}}{\text{s}}}$$

$$v_C = r_{C/IC}\omega_{bar} = \frac{L}{2\tan\theta}\omega_{bar} \qquad \boxed{\mathbf{v}_C = 0.433\mathbf{i}\,\frac{\text{m}}{\text{s}}}$$

$$v_D = r_{D/IC}\omega_{bar} \qquad \mathbf{v}_D = v_D(\cos(\phi)\mathbf{i} - \sin(\phi)\mathbf{j})$$

We need to determine $r_{D/IC}$ and the angle ϕ before we can calculate the velocity of point D.

$$r_{D/IC}^2 = r_{C/IC}^2 + (L/4)^2 = \left(\frac{L}{2\tan\theta}\right)^2 + (L/4)^2 = \frac{L^2}{4\tan^2\theta} + \frac{L^2}{16} = \frac{L^2(4+\tan^2\theta)}{16\tan^2\theta}$$

$$r_{D/IC} = 0.9014\,\text{m} \qquad v_D = 0.45\,\frac{\text{m}}{\text{s}}$$

$$\tan\phi = \frac{(L/4)}{r_{C/IC}} = \frac{\tan\theta}{2} \qquad \phi = 16.1^o$$

$$\boxed{\mathbf{v}_D = 0.432\mathbf{i} - 0.125\mathbf{j}\,\frac{\text{m}}{\text{s}}}$$

The IC for the wheels is located where the wheel touches the ground, provided that wheel does not slip. We can use this fact to determine the angular velocity of the wheel.

$$v_A = r\omega_{wheel} \qquad \omega_{wheel} = \frac{v_A}{r} \qquad \boxed{\omega_{wheel} = 2.94\,\frac{\text{rad}}{\text{s}}}$$

4.5) RELATIVE SLIDING IN MECHANISMS

If a mechanism, such as that shown in Figure 4.5-1, has a component on one link that slides relative to another, the analysis takes on one more layer of complexity. In some mechanisms, it is necessary to express the motion of a single point in two different ways, that is, with respect to two different points whose motion is known. The mechanism shown in Figure 4.5-1 may be analyzed in a similar manner. However, care must be taken because the two bodies that make up the mechanism are allowed to slide relative to one another.

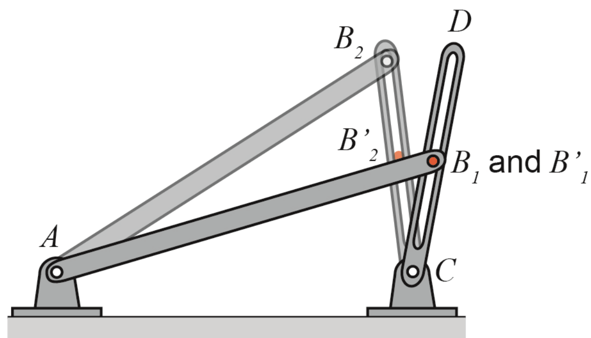


Figure 4.5-1: Mechanism with a sliding contact

The mechanism shown in Figure 4.5-1 has a sliding contact at end B. In order to analyze the motion of this mechanism, we need to express the motion of point B in two different ways. Specifically, we need to describe the motion of point B relative to the fixed point A and then relative to fixed point C. Relating the velocity and acceleration relative to fixed point A is relatively straightforward as shown in the following relations.

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$
$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$

How to express the motion of point B relative to the fixed point C may not be as clear. To help understand this situation better, imagine that there is a point B' that is attached to bar CD and is coincident with point B at a given instant. The motion of point B' can be expressed relative to point C in a similar manner to the relationships given above. However, point B (fixed to bar AB and sliding relative to bar CD), and B' (fixed to bar CD) do not necessarily have the same velocity or acceleration. This becomes clear by examining Figure 4.5-1 and realizing that even though points B and B' coincide at state 1, they do not at state 2, which means that they must have different velocities. Expressions for the motion of point B with respect to point C must include a term for the motion of point B relative to the motion of point B' as shown in Equation 4.5-1.

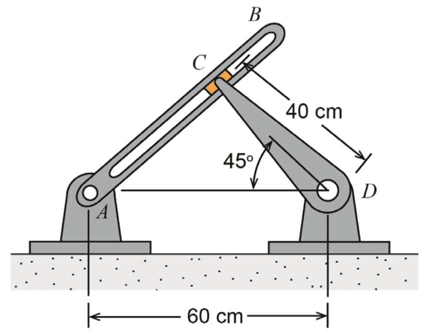
Velocity with relative sliding: $$\boxed{\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{B/C} + \mathbf{v}_{B/B'}} \quad (4.5\text{-}1)$$

$\mathbf{v}_B$ = linear velocity of point B
$\mathbf{v}_C$ = linear velocity of point C
$\boldsymbol{\omega}_{CD}$ = angular velocity of bar CD
$\mathbf{r}_{B/C}$ = position of point B (and B') relative to point C
$\mathbf{v}_{B/B'}$ = velocity of point B' relative to point B

The magnitude of the velocity of point B relative to point B' is unknown, but its direction may be determined from the geometry of the problem. It is known that vector $\mathbf{v}_{B/B'}$ is directed along the length of the arm CD. The equations for acceleration are not as straightforward since bar CD is rotating, therefore, the acceleration of point B relative to point B' may have a component that is not directed along bar CD. The motion of point B relative to point B' can more generally be understood as the motion of a particle relative to a rotating reference frame. This situation is described in detail in the section that follows.

Example Problem 4.5-1

Consider the mechanism shown. Arm *CD* is rotating at 4 rad/sec in the counterclockwise direction and block *C* slides in the slot of bar *AB*. Determine the angular velocity of bar *AB* for the instant shown.



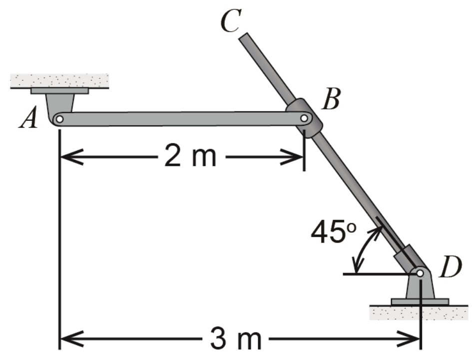
Given:

Find:

Solution:

Solved Problem 4.5-2

A collar is pinned to the end of rod AB. The collar pivots about point B and is allowed to slide freely along rod CD. If rod CD is given a counterclockwise angular velocity of 3 rad/sec, determine the velocity of point B ($\mathbf{v}_B$) for the instant shown in the given figure.

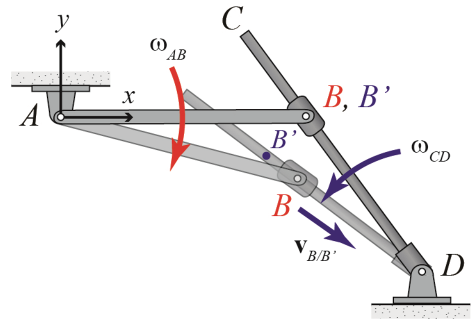


Given: $\omega_{CD} = 3$ rad/sec (ccw)

Find: $\mathbf{v}_B$

Solution:

Step 1: Get familiar with the system.

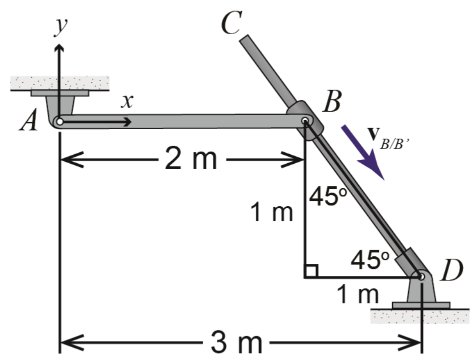


- We will assume a direction for ω_{AB} (cw) and will create a point B' on bar CD that is coincident with point B for the instant shown. From inspection, we can see that the velocity of point B relative to B' will be directed along rod CD.
- Also note that points A and D are fixed and their velocities are zero.

Step 2: Analyze the velocities.

Our approach for analyzing the velocities will be to find two different expressions for the velocity of point B. We will then set them equal and solve for the unknowns. Each of the two expressions will start from points of known velocity, that is, point A and point D.

Starting from point A:



$$\mathbf{v}_B = \cancel{\mathbf{v}_A} + \mathbf{v}_{B/A} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$
$$= -\omega_{AB}\,\mathbf{k} \times 2\mathbf{i}$$
$$= -2\omega_{AB}\mathbf{j} \qquad (1)$$

Similarly, starting from point D:

$$\mathbf{v}_B = \cancel{\mathbf{v}_D} + \mathbf{v}_{B'/D} + \mathbf{v}_{B/B'} = \boldsymbol{\omega}_{CD} \times \mathbf{r}_{B'/D} + \mathbf{v}_{B/B'} = 3\mathbf{k} \times (-1\mathbf{i} + 1\mathbf{j}) + v_{B/B'}(\cos 45\mathbf{i} - \sin 45\mathbf{j})$$
$$= (-3 + \frac{v_{B/B'}\sqrt{2}}{2})\mathbf{i} + (-3 - \frac{v_{B/B'}\sqrt{2}}{2})\mathbf{j} \qquad (2)$$

If we set Equation (1) and Equation (2) equal, we have two unknowns (ω_{AB} and $v_{B/B'}$) which can then be solved from between the x- and y-direction equations.

$$\mathbf{i}: 0 = -3 + \frac{v_{B/B'}\sqrt{2}}{2}$$

$$\mathbf{j}: -2\omega_{AB} = -3 - \frac{v_{B/B'}\sqrt{2}}{2}$$

In this instance, only the x-direction equation is needed to solve for $v_{B/B'}$.

$$\Rightarrow v_{B/B'} = \frac{6}{\sqrt{2}} = 3\sqrt{2}\,\frac{\text{m}}{\text{s}}$$

Substituting back into Equation (2) gives us the solution for v_B.

$$\mathbf{v}_B = \left(-3 + \frac{(3\sqrt{2})\sqrt{2}}{2}\right)\mathbf{i} + \left(-3 - \frac{(3\sqrt{2})\sqrt{2}}{2}\right)\mathbf{j}$$

$$\boxed{\mathbf{v}_B = -6\mathbf{j}\,\frac{\text{m}}{\text{s}}}$$

For good measure, we can also find that ω_{AB} equals 3 rad/sec.

4.6) ROTATING REFERENCE FRAME

4.6.1) POSITION

So far in this chapter we have described the general planar motion of rigid bodies with respect to reference frames that *do not rotate*. It may also be desirable to consider a body's motion in a frame that is rotating, for example, in terms of a polar coordinate frame. Consider the case of general planar motion shown in Figure 4.6-1 where the body-fixed x-y axes are allowed to rotate as well as translate. The position of point A in the body-fixed coordinate system attached to point B is represented by the unit direction vectors $\mathbf{i}$ and $\mathbf{j}$. This is the same equation that was seen when the coordinate system did not rotate.

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B} = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j})$$

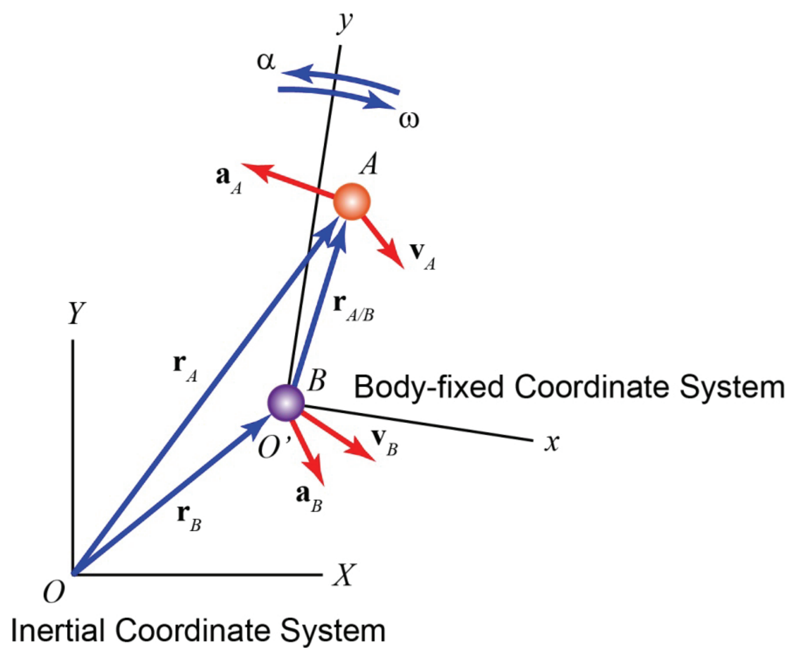


Figure 4.6-1: Motion relative to a rotating reference frame

4.6.2) VELOCITY

To determine the absolute velocity of point A we can differentiate the position equation to generate the following expression. Note that since the body-fixed coordinate frame is rotating, the derivatives of the unit-direction vectors may be non-zero.

$$\mathbf{v}_A = \frac{d}{dt}\left(\mathbf{r}_B + (x\mathbf{i} + y\mathbf{j})\right) = \mathbf{v}_B + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j}) + (x\dot{\mathbf{i}} + y\dot{\mathbf{j}})$$

If the x-y frame is non-rotating, the above reduces to $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$. This is the equation that we derived earlier in this chapter. In this case, however, the x-y frame is rotating and hence the unit vectors may not have derivatives equal to zero. This was the situation we encountered when considering the n-t and polar coordinate frames which also are allowed to rotate. Employing the same logic used in the sections on *n-t* and polar coordinates, the derivative of $\mathbf{i}$ can be shown to equal $\omega\mathbf{j}$, where ω is the angular velocity with which the *x-y* coordinate frame is rotating. Likewise, the derivative of $\mathbf{j}$ can be shown to equal $-\omega\mathbf{i}$. These relations expressed employing the cross product are shown below.

$$\dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i} \quad \text{and} \quad \dot{\mathbf{j}} = \boldsymbol{\omega} \times \mathbf{j}$$

Substituting the above expressions into the previous velocity equation, we get the following relationship for the absolute velocity of point A in terms of the rotating reference frame.

$$\begin{aligned} \mathbf{v}_A &= \mathbf{v}_B + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j}) + (x\dot{\mathbf{i}} + y\dot{\mathbf{j}}) \\ &= \mathbf{v}_B + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j}) + x(\boldsymbol{\omega} \times \mathbf{i}) + y(\boldsymbol{\omega} \times \mathbf{j}) \\ &= \mathbf{v}_B + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j}) + \boldsymbol{\omega} \times (x\mathbf{i} + y\mathbf{j}) \end{aligned}$$

Rewriting the preceding equation one more time, we get Equation 4.6-1 for the absolute velocity of point A.

Velocity in a rotating reference frame: $$\boxed{\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B} + \mathbf{v}_{A,rel}} \quad (4.6\text{-}1)$$

$\mathbf{v}_A$, $\mathbf{v}_B$ = absolute velocity of points A and B respectively
$\mathbf{v}_{A,rel}$ = velocity of point A relative to the rotating reference frame
ω = angular velocity of the rotating reference frame (rad/s)
$\mathbf{r}_{A/B}$ = position of point A relative to point B

In Equation 4.6-1, the term $\boldsymbol{\omega} \times \mathbf{r}_{A/B} + \mathbf{v}_{A,rel}$ represents the velocity of point A relative to point B, that is, $\mathbf{v}_{A/B}$. Intuitively, $\mathbf{v}_{A,rel}$ is the velocity of point A as seen by someone rotating with the body-fixed *x-y* frame, and $\boldsymbol{\omega} \times \mathbf{r}_{A/B}$ is due to the rotation of this coordinate frame. The $\mathbf{v}_{A,rel}$ term can also be thought of as the velocity of point A relative to a point A' which is attached to the rotating reference frame but is coincident with point A at the given instant. This was the situation we explored in the previous section when we considered mechanisms with a relative sliding contact. The above derivation assumed all terms were expressed in the rotating reference frame (*x-y*). All of the terms do not need to be expressed in the *x-y* frame, but if you wish to combine terms, then they need to be converted to a consistent frame.

In order to better understand the general expression given in Equation 4.6-1, it may be helpful to consider a specific case of a rotating reference frame we have already encountered. In the case of polar coordinates, $\mathbf{v}_{A,rel}$ is equal to $\dot{r}\mathbf{e}_r$ which is the instantaneous speed of the particle as seen by someone rotating with the polar axes. The term $\boldsymbol{\omega} \times \mathbf{r}_{A/B}$ is equal to $r\dot{\theta}\mathbf{e}_\theta$ and is due to the rotation of the polar axes.

4.6.3) ACCELERATION

To determine the absolute acceleration of point A we can differentiate Equation 4.6-1 to generate the following expression.

$$\begin{aligned}\mathbf{a}_A &= \frac{d}{dt}\left(\mathbf{v}_B + \mathbf{v}_{A,rel} + \boldsymbol{\omega}\times\mathbf{r}_{A/B}\right)\\ &= \mathbf{a}_B + \dot{\mathbf{v}}_{A,rel} + \dot{\boldsymbol{\omega}}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times\dot{\mathbf{r}}_{A/B}\end{aligned}$$

Recognizing that $\mathbf{v}_{A,rel}$ and $\mathbf{r}_{A/B}$ are expressed in the rotating x-y reference frame, their time derivatives will include derivatives of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ as described earlier. Therefore, the above equation is equal to the following.

$$\mathbf{a}_A = \mathbf{a}_B + (\boldsymbol{\omega}\times\mathbf{v}_{A,rel} + \mathbf{a}_{A,rel}) + \dot{\boldsymbol{\omega}}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/B} + \mathbf{v}_{A,rel})$$

Rearranging and combining like terms, we get Equation 4.6-2 for the absolute acceleration of point A with respect to the primary non-rotating reference frame. Again referring back to the case of polar coordinates, $\mathbf{a}_{A,rel}$ is equal to $\ddot{r}\mathbf{e}_r$, 2ω x $\mathbf{v}_{A,rel}$ is equal to $2\dot{r}\dot{\theta}\mathbf{e}_\theta$, α x $\mathbf{r}_{A/B}$ is equal to $r\alpha\mathbf{e}_\theta$ and ω x (ω x $\mathbf{r}_{A/B}$) is equal to $-r\dot{\theta}^2\mathbf{e}_r$. If, however, particle A is traveling a curved path in the rotating reference frame, then $\mathbf{a}_{A,rel}$ won't be equal to $\ddot{r}\mathbf{e}_r$, it will have a normal and tangential component.

Acceleration in a rotating reference frame:

$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} = \mathbf{a}_B + 2\boldsymbol{\omega}\times\mathbf{v}_{A,rel} + \boldsymbol{\alpha}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/B}) + \mathbf{a}_{A,rel}} \quad (4.6\text{-}2)$$

or

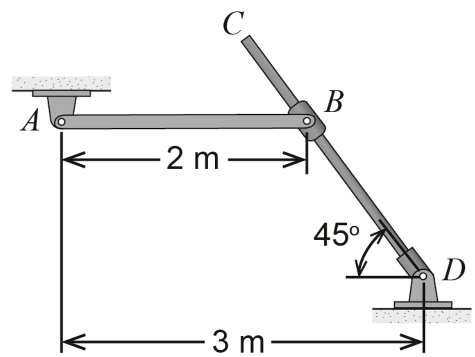
$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} = \mathbf{a}_B + 2\boldsymbol{\omega}\times\mathbf{v}_{A,rel} + \boldsymbol{\alpha}\times\mathbf{r}_{A/B} - \omega^2\mathbf{r}_{A/B} + \mathbf{a}_{A,rel}}$$

$\mathbf{a}_A$, $\mathbf{a}_B$ = absolute acceleration of points A and B respectively
$\mathbf{v}_{A,rel}$ = velocity of point A relative to the rotating reference frame
$\mathbf{a}_{A,rel}$ = acceleration of point A relative to the rotating reference frame
ω = angular velocity of the rotating reference frame (rad/s)
α = angular acceleration of the rotating reference frame (rad/s^2)
$\mathbf{r}_{A/B}$ = position of point A relative to point B

Another type of body-fixed reference frame that we have employed is the n-t coordinate system. Whereas the relative velocity and acceleration equations in polar coordinates have the exact form of Equations 4.6-1 and 4.6-2, the relative motion equations in n-t coordinates do not. This follows from the fact the polar coordinate frame is defined in terms of coordinates r and θ that are defined with respect to another reference frame. The n-t coordinate frame is not defined with respect to another reference frame. It is defined in terms of the particle's path (i.e. the particle's speed v along the path of travel), and the path's radius of curvature ρ.

Solved Problem 4.6-1

A collar is pinned to the end of rod AB. The collar pivots about point B and is allowed to slide freely along rod CD. If rod CD is given a counterclockwise angular velocity of 3 rad/sec that is decreasing at a rate of 1 rad/sec^2, determine the velocity $\mathbf{v}_B$ and acceleration $\mathbf{a}_B$ of point B for the instant shown in the given figure.



Given: $\omega_{CD} = 3$ rad/sec (ccw)
$\alpha_{CD} = 1$ rad/sec^2 (cw)

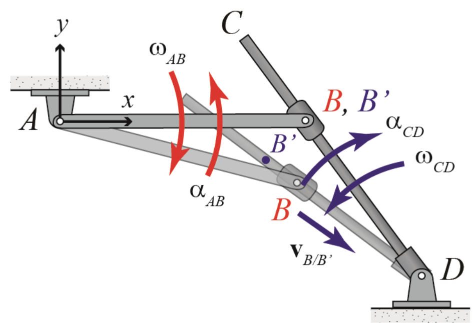
Find: $\mathbf{v}_B$, $\mathbf{a}_B$

Solution:

This system is the same one we considered in Solved Problem 4.5-2. We will revisit the problem, but will solve it in terms of a rotating reference x-y frame attached to bar CD. We will also determine the acceleration of point B in this case.

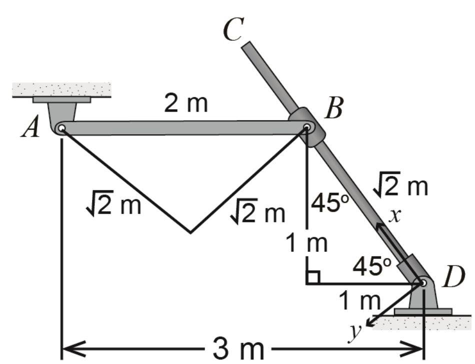
Step 1: Get familiar with the system.

- We will again assume a direction for ω_{AB} (cw) and α_{AB} (ccw). From inspection, we can determine that the direction of $\mathbf{v}_{B,rel}$ and $\mathbf{a}_{B,rel}$ (the velocity and acceleration of point B relative to the rotating reference frame x-y) is along the bar CD. Note: If the rod CD were curved, then $\mathbf{a}_{B,rel}$ would also have a component normal to the rod.



Step 2: Analyze the velocities.

We will use the same basic approach from Solved Problem 4.5-2 where we started from two known velocities (for point A and point D) and met in the middle at point B.



Starting from point A:

$$\mathbf{v}_B = \cancel{\mathbf{v}_A} + \mathbf{v}_{B/A} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = -\omega_{AB}\,\mathbf{k} \times (-2\cos 45\mathbf{i} - 2\sin 45\mathbf{j})$$
$$= -\omega_{AB}\sqrt{2}\mathbf{i} + \omega_{AB}\sqrt{2}\mathbf{j} \qquad (1)$$

Similarly, starting from point D:

$$\mathbf{v}_B = \cancel{\mathbf{v}_D} + \mathbf{v}_{B/D} = \boldsymbol{\omega}_{CD} \times \mathbf{r}_{B/D} + \mathbf{v}_{B,rel} = (3\mathbf{k} \times \sqrt{2}\mathbf{i}) - v_{B,rel}\mathbf{i}$$
$$= -v_{B,rel}\mathbf{i} + 3\sqrt{2}\mathbf{j} \qquad (2)$$

If we set Equation (1) and Equation (2) equal we have two unknowns (ω_{AB} and $v_{B,rel}$) which can then be solved from between the x- and y-direction equations.

$$\mathbf{i}: -\omega_{AB}\sqrt{2} = -v_{B,rel}$$

$$\mathbf{j}: \omega_{AB}\sqrt{2} = 3\sqrt{2}$$

Simultaneously solving the two above equations provides that ω_{AB} = 3 rad/sec and $v_{B,rel}$ = $3\sqrt{2}$ m/s. Then substituting back into either Equation (1) or Equation (2) provides the expression for $\mathbf{v}_B$. Notice that each of these values is consistent with the results from Solved Problem 4.5-2. The only difference is that $\mathbf{v}_B$ is represented in a different coordinate frame.

$$\boxed{\mathbf{v}_B = -3\sqrt{2}\mathbf{i} + 3\sqrt{2}\mathbf{j}\ \frac{\text{m}}{\text{s}}}$$

Step 3: Analyze the accelerations.

Here we will employ an approach that is similar to the one we applied in analyzing the velocities.

Starting from point A:

$$\mathbf{a}_B = \cancel{\mathbf{a}_A} + \mathbf{a}_{B/A} = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} + \boldsymbol{\omega}_{AB} \times (\boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A})$$
$$= \alpha_{AB}\ \mathbf{k} \times (-2\cos 45\mathbf{i} - 2\sin 45\mathbf{j}) + -3\mathbf{k} \times (-3\mathbf{k} \times (-2\cos 45\mathbf{i} - 2\sin 45\mathbf{j}))$$
$$= (\alpha_{AB}\sqrt{2} + 9\sqrt{2})\mathbf{i} + (-\alpha_{AB}\sqrt{2} + 9\sqrt{2})\mathbf{j} \qquad (3)$$

Similarly, starting from point D:

$$\mathbf{a}_B = \cancel{\mathbf{a}_D} + \mathbf{a}_{B/D} = 2\boldsymbol{\omega}_{CD} \times \mathbf{v}_{B,rel} + \boldsymbol{\alpha}_{cd} \times \mathbf{r}_{B/D} + \boldsymbol{\omega}_{CD} \times (\boldsymbol{\omega}_{CD} \times \mathbf{r}_{B/D}) + \mathbf{a}_{B,rel}$$
$$= (2(3\mathbf{k}) \times -3\sqrt{2}\mathbf{i}) + (-1\mathbf{k} \times \sqrt{2}\mathbf{i}) + (3\mathbf{k} \times (3\mathbf{k} \times \sqrt{2}\mathbf{i})) - a_{B,rel}\mathbf{i}$$
$$= (-9\sqrt{2} - a_{B,rel})\mathbf{i} - 19\sqrt{2}\mathbf{j} \qquad (4)$$

If we set Equation (3) and Equation (4) equal we have two unknowns (α_{AB} and $a_{B,rel}$) which can then be solved from between the x- and y-direction equations.

$$\mathbf{i}: \alpha_{AB}\sqrt{2} + 9\sqrt{2} = -9\sqrt{2} - a_{B,rel}$$

$$\mathbf{j}: -\alpha_{AB}\sqrt{2} + 9\sqrt{2} = -19\sqrt{2}$$

Simultaneously solving the two above equations provides that α_{AB} = 28 rad/sec^2 and $a_{B,rel} = -46\sqrt{2}$ m/s^2. The negative sign on $a_{B,rel}$ means that it is actually oriented opposite the direction we originally assumed. Substituting this value back into either Equation (3) or Equation (4) provides the expression for $\mathbf{a}_B$.

$$\boxed{\mathbf{a}_B = 37\sqrt{2}\mathbf{i} - 19\sqrt{2}\mathbf{j}\ \frac{\mathrm{m}}{\mathrm{s}^2}}$$

CHAPTER 4 REVIEW PROBLEMS

RP4-1) Explain the difference between relative and absolute velocities.

RP4-2) Consider a rigid body undergoing pure rotation. Every point on that body moves in a ____________ path around the fixed axis.

RP4-3) Points on a body undergoing pure rotation that are at different distances from the fixed axis have different velocities. (True, False)

RP4-4) Points on a body undergoing pure rotation that are at different distances from the fixed axis have different angular velocities. (True, False)

RP4-5) Rigid-body motion has two components. What are they?

RP4-6) The velocity of any point on a body experiencing fixed-axis rotation is $v_A = r\omega$. What direction is this velocity?

RP4-7) Consider the velocity equation for point B on a rigid body ($\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$). Which component accounts for the translation of the body and which accounts for the rotation of the body about A?

RP4-8) How can the *instantaneous center of zero velocity* make our calculations easier?

RP4-9) The velocity of a point on a rigid body is always perpendicular to the line that connects the point to the _______________________.

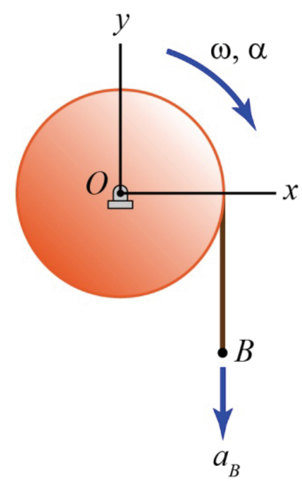
RP4-10) Consider two points (A, B) on a rigid body and the instantaneous center of zero velocity (IC). If point A is twice as far away from the IC as B is, write an equation relating v_A to v_B.

RP4-11) A wheel of radius r is rolling while slipping at the point of contact between the wheel and ground. The speed of the wheel's center is $v = r\omega$. (True, False)

RP4-12) Consider the equation for the acceleration of point B on a rigid body ($\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$). Which component accounts for the translation of the body, which component accounts for the tangential acceleration and which component accounts for the normal acceleration?

RP4-13) The acceleration of an IC is generally zero. (True, False)

RP4-14) A cord is wrapped around a cylinder of radius 0.2 meters, as shown in the figure. Starting from rest, the cord is pulled downward with a constant acceleration of 10 m/s^2. Determine the angular acceleration and angular velocity of the disk after it has completed 10 revolutions.



Given: a_B = 10 m/s^2 = constant
ω_o = 0
θ_f = 10 revolutions
r = 0.2 m

Find: α_f, ω_f

Solution:

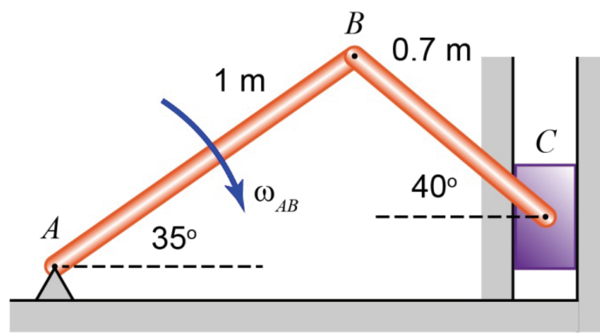
Find the angular acceleration.

α = ______________________

Find the final angular velocity.

ω = ______________________

RP4-15) When the slider block C is in the position shown, the link AB has a clockwise angular velocity of 2 rad/s. Determine the velocity of block C at this instant. The length of link AB is 1 meter and the length of link BC is 0.7 meters.



Given: ω_{AB} = 2 rad/s
l_{AB} = 1 m
l_{BC} = 0.7 m

Find: $\mathbf{v}_C$

Solution:

Assign a direction for the angular velocity of link BC.

What direction is the velocity of C?

$\mathbf{v}_C = v_C$ (?)

Find the velocity of B.

$\mathbf{v}_B$ = ______________________

Find the velocity of C as a function of ω_{BC}.

$\mathbf{v}_C(\omega_{BC})$ = ______________________

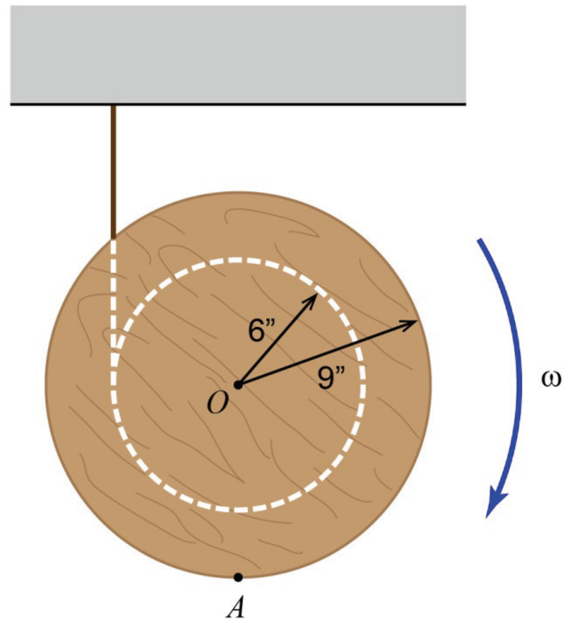
Find the angular velocity of link BC.

ω_{BC} = ______________________

Find the velocity of C.

$\mathbf{v}_C$ = ______________________

RP4-16) A cord is wrapped around the inner hub of a wooden spool. The end of the cord is attached to a fixed horizontal support. The spool is released and allowed to fall under the influence of gravity as the cord unwinds. Assuming that the spool does not sway back and forth, determine the velocity of point A for the instant shown where the speed of the center of the spool (point O) has reached 5 ft/s.



Given: v_O = 5 ft/s
r_i = 6 in, r_o = 9 in

Find: $\mathbf{v}_A$

Solution:

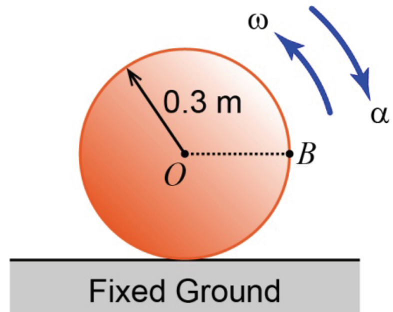
Find the angular velocity.

ω = ______________________

Find the velocity of point A.

$\mathbf{v}_A$ ______________________

RP4-17) The ball shown rolls without slipping and has an angular velocity and angular acceleration of 6 rad/s and 4 rad/s^2, respectively, with the directions shown in the figure. Determine the acceleration of point *B* at this instant.



Given: ω = 6 rad/s
α = 4 rad/s^2
r = 0.3 m

Find: $\mathbf{a}_B$

Solution:

Locate the IC.

$v_{IC} = 0$ True False
$a_{IC} = 0$ True False

Find the acceleration of *O*.

$\mathbf{a}_O$ = ______________________

Find the acceleration of *B*.

$\mathbf{a}_B$ = ______________________

Explain the differences between $\mathbf{a}_O$ and $\mathbf{a}_B$.

i-component:

j-component:

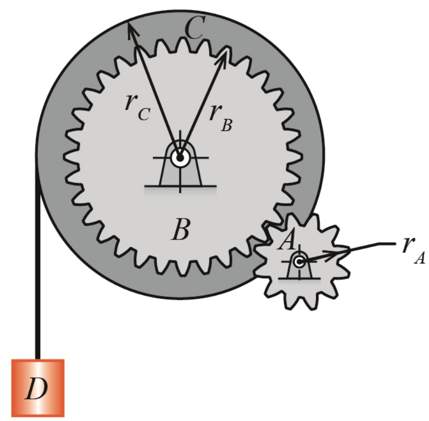
CHAPTER 4 PROBLEMS

P4.1) BASIC FIXED-AXIS ROTATION PROBLEMS

P4.1-1) A circular saw blade is rotating at 1500 rev/min when power to the saw is turned off. The angular speed of the blade decelerates at a rate of $\alpha = 2t$ rad/s^2 where t is in seconds. Determine the number of revolutions it takes for the saw blade to come to rest.

Ans: θ = 209 rev

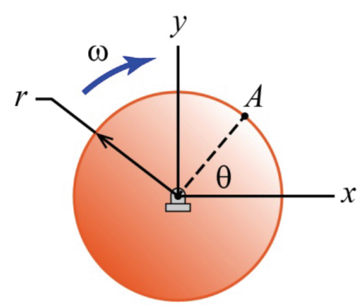
P4.1-2)^{fe} It is desired that the shown hoisting mechanism be operated such that the load D is lifted at a constant rate of 2 ft/s. If the drum C, which is rigidly attached to gear B, has a radius of r_C = 4 ft and the gear ratio between B and A is 3:1, determine the angular velocity with which a motor must drive pinion gear A.



a) ω = 1.5 rad/s ccw   b) ω = 1.5 rad/s ccw

c) ω = 1.5 rad/s ccw   d) ω = 1.5 rad/s ccw

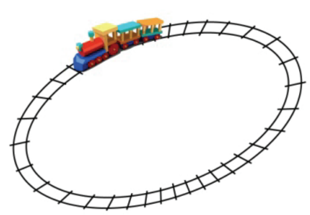
P4.1-3)^{fe} A 20-in diameter disk rotates about its center point with a constant angular velocity of 100 rpm in the clockwise direction. What is the velocity of point A on its rim if θ = 25°? Solve for the velocity the x-y coordinate systems.



a) $\mathbf{v}_A = 44.2\mathbf{i} - 94.9\mathbf{j}$ in/s   b) $\mathbf{v}_A = 44.2\mathbf{i} - 94.9\mathbf{j}$ in/s

c) $\mathbf{v}_A = 44.2\mathbf{i} - 94.9\mathbf{j}$ in/s   d) $\mathbf{v}_A = 44.2\mathbf{i} - 94.9\mathbf{j}$ in/s

P4.1-4)^{fe} A toy train travels around a circular track. Its angular position is given by $\theta(t) = t^3 - 10t + 2$, where θ is given in radians and t is in seconds. Determine the train's angular acceleration when t = 10 seconds.

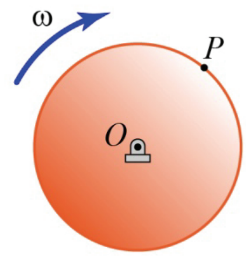


a) α = 580 rad/s^2 b) α = 580 rad/s^2

c) α = 580 rad/s^2 d) α = 580 rad/s^2

P4.2) INTERMEDIATE FIXED-AXIS ROTATION PROBLEMS

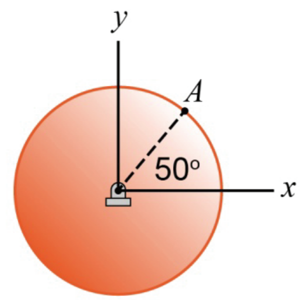
P4.2-1)[fe] A circular disk of radius 0.5 m rotates about its center O in the direction shown with a constant angular acceleration. Point P on the rim has an acceleration of $\mathbf{a}_P = 2\mathbf{e}_t + 5\mathbf{e}_n$ m/s^2 at t = 0. Determine the number of revolutions made by the disk in 5 seconds.



a) θ = 10.5 rev b) θ = 10.5 rev

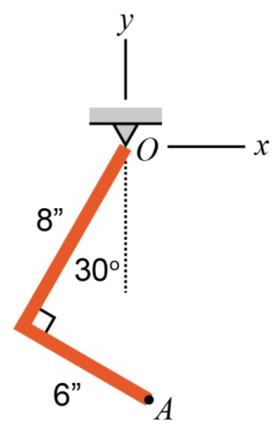
c) θ = 10.5 rev d) θ = 10.5 rev

P4.2-2) A circular disk of radius 0.2 m rotates about its center. The acceleration of point A on the rim of the disk is $\mathbf{a} = 5\mathbf{i} - 7\mathbf{j}$ m/s^2. Determine the angular velocity and angular acceleration of the disk at this instant.



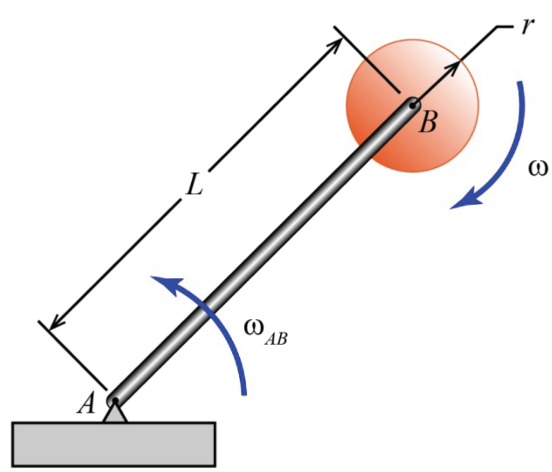
Ans: $\boldsymbol{\omega}$ = 3.28 $\mathbf{k}$ rad/s, $\boldsymbol{\alpha}$ = 41.6 $\mathbf{k}$ rad/s^2

P4.2-3) The angled pendulum is swinging freely about the fixed axis perpendicular to the vertical plane that passes through O. On its upswing, and at the moment when its long arm makes an angle of 30 degrees with the vertical, the pendulum's angular velocity is 3 rad/s clockwise and its angular acceleration is 4 rad/s^2 counter-clockwise. Determine the velocity and acceleration of the tip of the pendulum (point A) at this instant.



Ans: $\mathbf{v}_A = -2.48\mathbf{i} - 0.3\mathbf{j}$ ft/s, $\mathbf{a}_A = 2.41\mathbf{i} + 7.84\mathbf{j}$ ft/s^2

P4.2-4) A bar and disk rotate as a unit with an angular velocity of ω_{AB} = 4 rad/s in the direction shown. The disk also rotates independently with an angular velocity of ω = 10 rad/s in the direction shown. If the radius of the disk is 10 cm and the length of the bar is 80 cm, determine the location on the body with the greatest speed and determine the value of that speed.

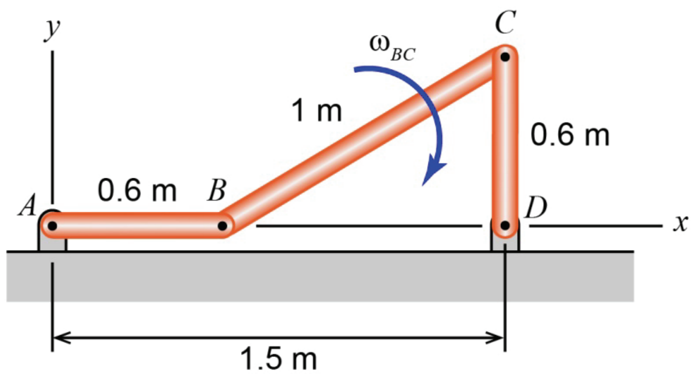


Ans: v = 420 cm/s

P4.3) BASIC GENERAL PLANAR MOTION PROBLEMS

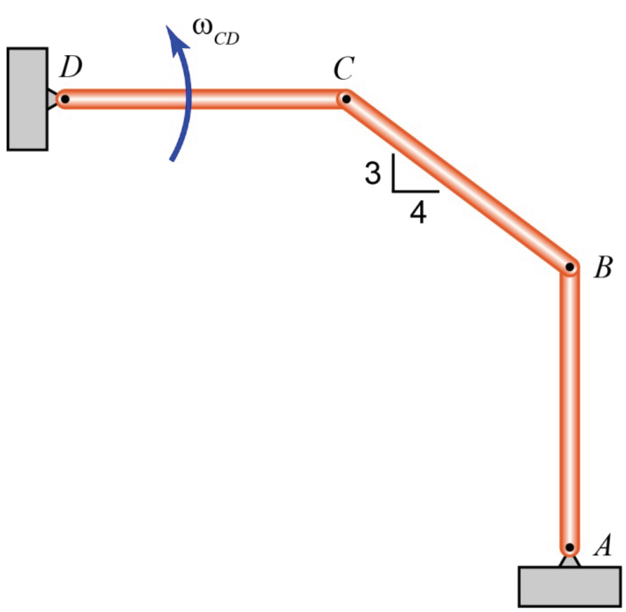
P4.3-1) At the instant shown in the figure, link BC has a clockwise angular velocity of ω_{BC} = -2 $\mathbf{k}$ rad/s. Determine the angular velocity of links AB and CD.

Ans: ω_{AB} = 3 $\mathbf{k}$ rad/s,
ω_{CD} = -2 $\mathbf{k}$ rad/s



P4.3-2)[fe] The 3-bar linkage shown is set in to motion by applying a counter clockwise angular velocity of 10 rad/s to bar CD. All links have the same length. Determine the angular velocity of bar AB at the instant represented in the figure.

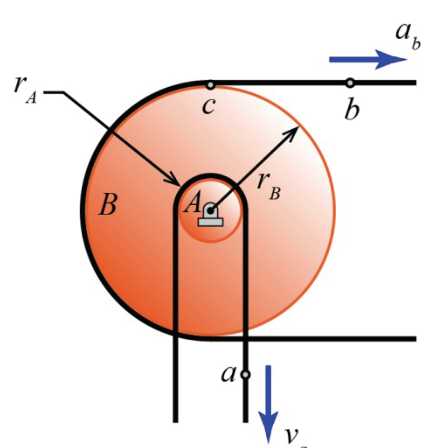
Ans: ω = 7.5 rad/s



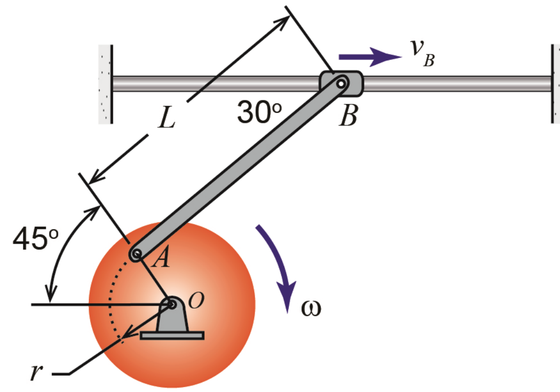
P4.4) INTERMEDIATE GENERAL PLANAR MOTION PROBLEMS

P4.4-1) The belt driven pulley system shown has a smaller pulley (r_A = 3 in) rigidly attached to a larger pulley (r_B = 12 in). The belt velocity of the smaller pulley is 4 ft/s and the belt acceleration of the larger pulley is 0.5 ft/s^2. Determine the angular velocity and acceleration of the pulley system. Also, determine the belt speed of the larger pulley and the acceleration of point c located on the rim of the larger pulley.

Ans: ω = 16 rad/s, α = 0.5 rad/s^2, $v_b = 16$ ft/s,
$\mathbf{a}_c = 0.5\mathbf{e}_t + 256\mathbf{e}_n$ ft/s^2



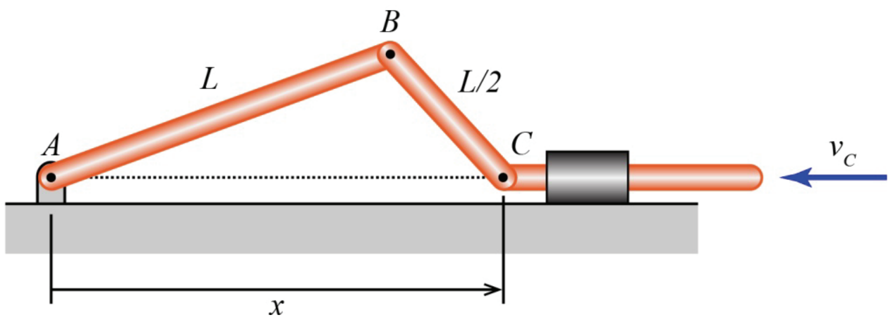
P4.4-2) The disk shown is pinned at O and is rotating clockwise with an angular velocity of ω = 5 rad/sec. The disk is connected via rod AB (L = 4 m) to collar B. The A end of rod AB is attached to the disk at a distance of r = 0.8 m from point O. Collar B is constrained to move along a horizontal guide. At the instant shown, determine the velocity of point B.



Ans: v_B = 4.46 m/s

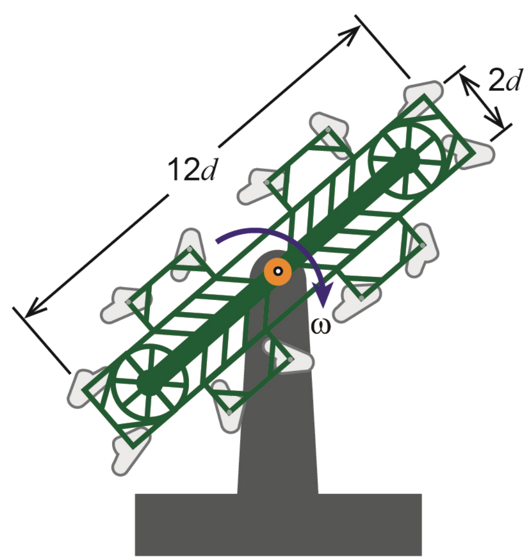
P4.5) ADVANCED GENERAL PLANAR MOTION PROBLEMS

P4.5-1) The drive link in the mechanism shown runs in a linear bearing with a velocity v_C directed to the left. Determine the angular velocity of link AB and BC as a function of v_C, x and L.



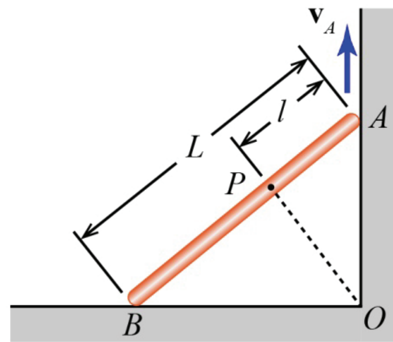
Ans: $\omega_{AB} = \frac{2v_C}{L}\left(\frac{2x_2}{L}\right)$

P4.5-2) You are designing the carnival ride shown and wish to prevent riders from experiencing accelerations greater than three times the acceleration due to gravity g. The cars attached to the main body rotate only due to gravity and may be assumed to experience a maximum angular acceleration of 10 rad/sec^2 and a maximum angular velocity of 4 rad/sec. Determine the limitations on the dimension d of the ride and on the constant angular velocity ω with which the main body of the ride can be driven with while still maintaining the 3g limit

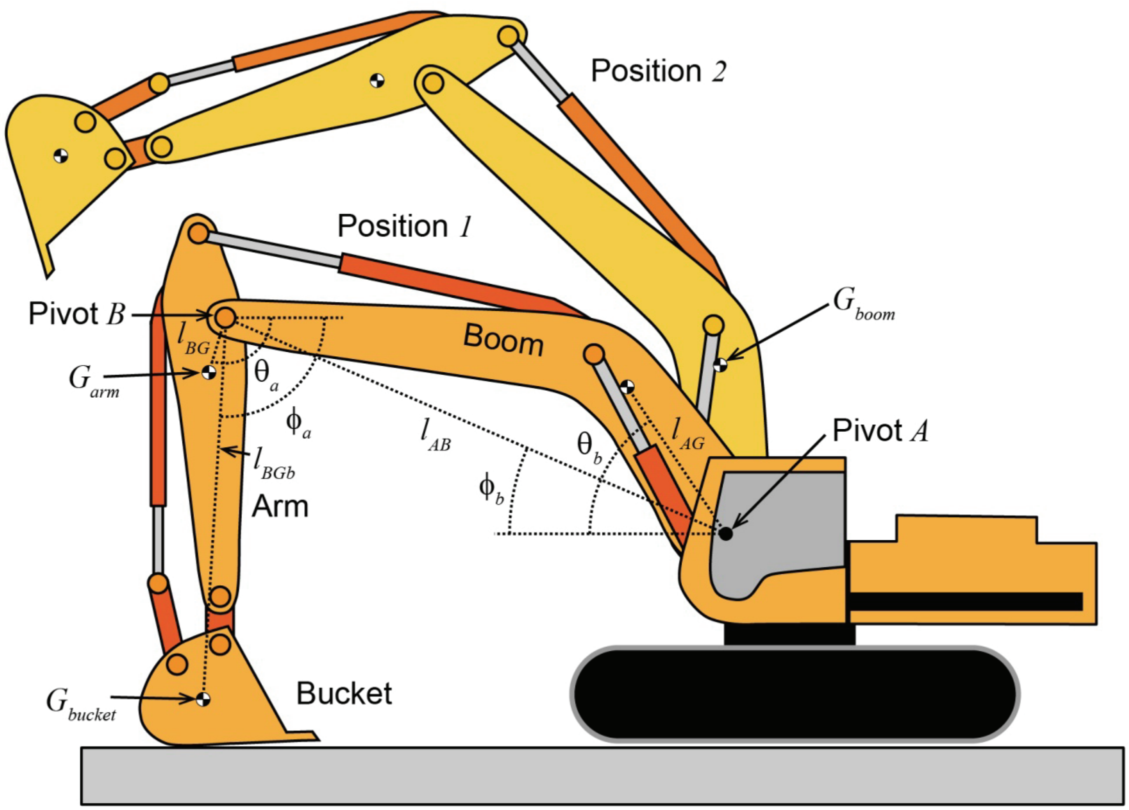


Ans: $\omega^2 d$ < 6.57 ft/s^2

P4.5-3) Rod AB contacts a wall and floor as shown in the figure. If these contacts are not broken and the velocity of end A is constant upward, prove that the acceleration of point P is always perpendicular to the line drawn from O to A.

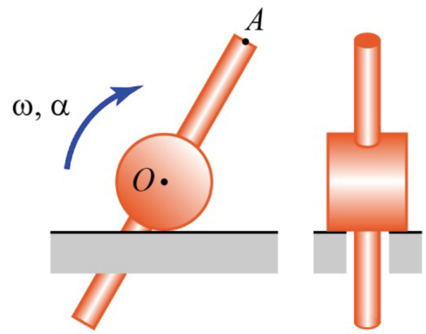


P4.5-4) The digging mechanism of an excavator consists of a boom, arm and bucket as shown in the figure. The boom rotates around pivot A which is fixed to the frame of the excavator. The arm rotates around pivot B. Determine the velocity of the bucket in terms of the boom and arm parameters. Important lengths and angles are given on the figure.



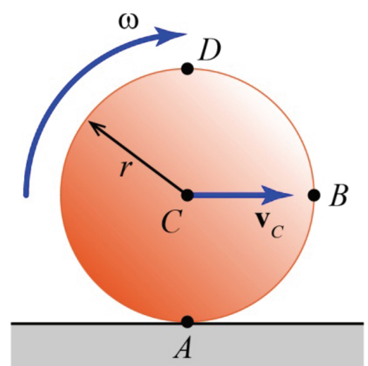
P4.6) BASIC IC & ROLLING PROBLEMS

P4.6-1) The wheel with a rigidly attached bar shown, rolls without slip. Determine the velocity and acceleration of the wheel center (Point O) and the end of the bar (Point A) when the bar is horizontal to the rolling surface. The radius of the wheel is 0.25 meters and the total length of the bar from end to end is 1 meter. At the instant the bar is horizontal, the angular velocity and angular acceleration directions are clockwise and have values of 4 rad/s and 5 rad/s^2 respectively.



Ans: $\mathbf{v}_O = \mathbf{i}$ m/s, $\mathbf{a}_O = 1.25\mathbf{i}$ m/s^2, $\mathbf{v}_A = \mathbf{i} - 2\mathbf{j}$ m/s, $\mathbf{a}_A = -6.75\mathbf{i} - 2.5\mathbf{j}$ m/s^2

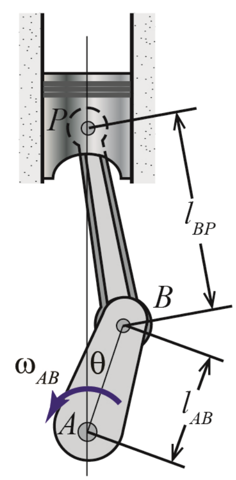
P4.6-2)[fe] A disk with radius r rolls without slipping on a horizontal surface. If the velocity of the disk's center is v_C, in the direction shown, determine the velocity of point B and D whose locations are identified on the figure.



a) $\mathbf{v}_B = r\omega(\mathbf{i} - \mathbf{j})$, $\mathbf{v}_D = 2\omega r\mathbf{i}$ b) $\mathbf{v}_B = r\omega(\mathbf{i} - \mathbf{j})$, $\mathbf{v}_D = 2\omega r\mathbf{i}$

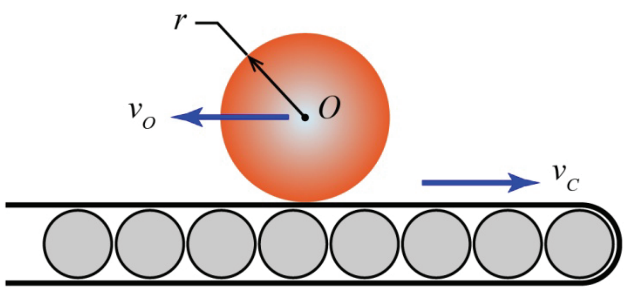
c) $\mathbf{v}_B = r\omega(\mathbf{i} - \mathbf{j})$, $\mathbf{v}_D = 2\omega r\mathbf{i}$ d) $\mathbf{v}_B = r\omega(\mathbf{i} - \mathbf{j})$, $\mathbf{v}_D = 2\omega r\mathbf{i}$

P4.6-3) Given the piston assembly shown, determine the location of the connecting rod's instantaneous center of zero velocity for the instant θ = 35°. The length of the connecting rod is l_{BP} = 10 in and the length of the crank shaft rod is l_{AB} = 4 in.



Ans: (9.11, 13) in

P4.6-4)[fe] A ball, of radius 2 meters, rides on a conveyer belt as shown in the figure. If the velocity of the ball's center O is 1.5 m/s, directed to the left, and the velocity of the top of the conveyer belt is 3 m/s, directed to the right, determine the angular velocity of the ball. Assume that the ball does not slip relative to the conveyer belt.



a) ω = 2.25 rad/s ccw b) ω = 2.25 rad/s ccw

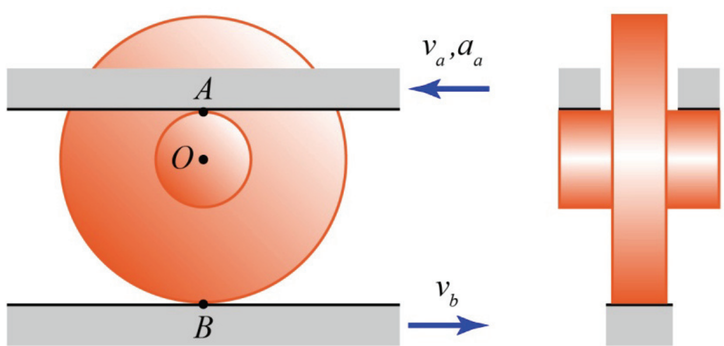
c) ω = 2.25 rad/s ccw d) ω = 2.25 rad/s ccw

P4.7) INTERMEDIATE IC & ROLLING PROBLEMS

P4.7-1) A car is driving on a road at 50 mph (22.4 m/s). The driver sees a squirrel and applies his brakes causing the car to decelerate at a constant rate of 4.5 m/s^2. The pavement is dry and the majority of the braking is done with the rear wheels such that the front wheels may be assumed to be rolling without slip. What is the velocity and acceleration of point A, on the car tire, 2 seconds after the driver applies the brakes? Point A is on the outer edge of the tire at the forward position that is level with the wheel's center at the end of this 2 seconds. The radius of the tire is 13 inches (0.33 m).

Ans: $\mathbf{v}_A = 13.4\mathbf{i} - 13.4\mathbf{j}$ m/s, $\mathbf{a}_A = -548.7\mathbf{i} - 4.5\mathbf{j}$ m/s^2

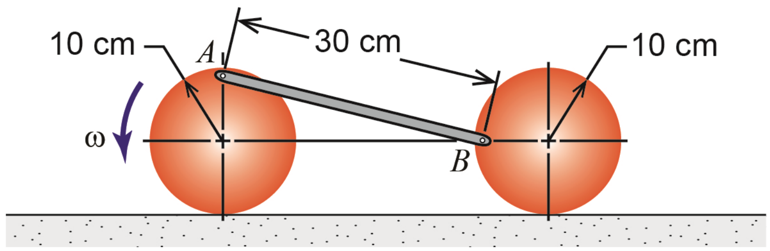
P4.7-2) A wheel is trapped and rolls without slip between two moving surfaces. The wheel has an inner hub and an outer hub. The radius of the inner hub is 6 inches and the radius of the outer hub is 18 inches. The velocity and acceleration of the surface touching the inner hub is 0.5 ft/s and 0.2 ft/s^2 respectively, in the direction shown, and the velocity of the surface touching the outer hub is a constant 2 ft/s, in the direction shown. Find the angular velocity and acceleration of the wheel and the velocity and acceleration of the wheel's center at this instant.



Ans: $\boldsymbol{\omega} = 1.25\,\mathbf{k}$ rad/s, $\boldsymbol{\alpha} = 0.1\,\mathbf{k}$ rad/s^2, $\mathbf{v}_O = 0.125\mathbf{i}$ ft/s, $\mathbf{a}_O = -0.15\mathbf{i}$ ft/s^2

P4.8) ADVANCED IC & ROLLING PROBLEMS

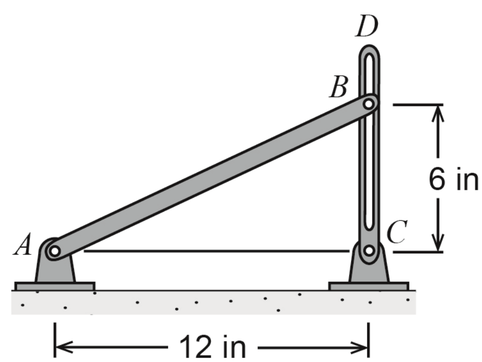
P4.8-1) Consider the two disks shown that are rolling without slip on level ground. If the disk on the left has an angular velocity of ω = 3 rad/sec in the counterclockwise direction, determine the angular velocity of the disk on the right for the instant shown.



Ans: ω = 9.28 rad/s ccw

P4.9) INTERMEDIATE RELATIVE SLIDING

P4.9-1) At the instant shown, bar AB has an angular velocity of ω_{AB} = 2 rad/sec in the counterclockwise direction that is decreasing at a rate of 3 rad/sec^2. Also, the pin at B slides in the slot of bar CD. Determine the angular velocity and angular acceleration of bar CD at this instant.



Ans: ω_{CD} = 2 rad/s ccw, α_{CD} = 5 rad/s ccw

CHAPTER 4 DESIGN PROBLEMS

D-1) Consider the "tea-cup" amusement park ride shown. Each cup has an occupancy limit of three riders. The cups are rotated with respect to the platform by the riders through a wheel at the center of the teacup. The platform itself is turned by a large electric motor. Use your knowledge of dynamics to address the following issues; a) it is desired that the sustained acceleration of a rider should be less than $3g$ and b) the peak acceleration experienced by a rider should be less than $5g$.

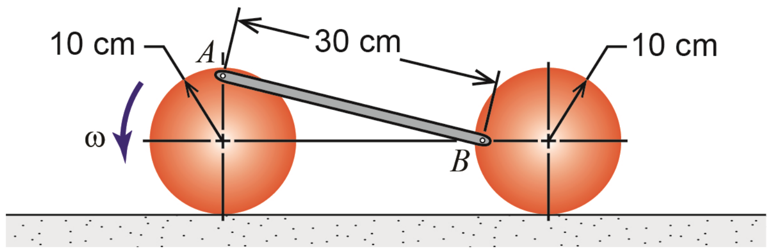
- Recommend the maximum angular speed with which the platform can rotate.
- Recommend the maximum angular acceleration with which the platform can rotate.

Note: There may be information that you need to complete your design which you do not have. Explain how you would obtained estimates of any missing information that you need.

CHAPTER 4 COMPUTER PROBLEMS

C-1) Perform a computer simulation of the motion of a particle on a rolling disk.

C-2) Consider the two disks shown that are rolling without slip on level ground. If the disk on the left has an angular velocity of ω = 3 rad/sec in the counterclockwise direction, plot the angular velocity of the disk on the right from the instant shown until point B makes a full revolution.



CHAPTER 4 ACTIVITIES

A-1) Ride on a merry-go-round and discuss why you "feel" different at different locations on the ride.

Group size: 1 to 2 students

Supplies:
- Stop watch
- Measuring tape

Procedure:
1. Find a merry-go-round. This may be a ride in an amusement park or one at a park that you need to spin.
2. Use the stop watch to determine the merry-go-round's angular velocity.
3. Mark and measure the radius of three different locations on the ride.
4. Ride the merry-go-round at each location for at least 30 seconds and make notes about how you "feel".

Data Sheet

Time per one revolution = ______________________

Angular velocity in rad/s = _____________________

	Location 1	Location 2	Location 3
Radius			
Velocity			
Acceleration			
"Feeling"			

Assumptions (List all your assumptions)